SPREADS IN PROJECTIVE HJELMSLEV SPACES OVER FINITE CHAIN RINGS

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ABSTRACT. We prove a necessary and sufficient condition for the existence of spreads in the projective Hjelmslev geometries $\mathrm{PHG}(R_R^{n+1})$. Further, we give a construction of projective Hjelmslev planes from spreads that generalizes the familiar construction of projective planes from spreads in $\mathrm{PG}(n,q)$.

Keywords: projective Hjelmslev plane, spread, arc, oval, finite chain ring

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1. INTRODUCTION

In this paper, we investigate spreads in the projective Hjelmslev geometries $\mathrm{PHG}(R_R^{n+1})$. There exists an extensive literature about spreads in the projective geometries $\mathrm{PG}(k,q)$ (cf. [6] and the references there). The same objects in the ring geometries have attracted little or no attention despite the connections to interesting areas as linear codes over finite chain rings.

In what follows, we restrict ourselves to spreads in geometries over chain rings of nilpotency index 2. Since geometries over rings have less regularities than the usual projective geometries, we settle for a problem that is tractable to some extent. On the other hand, this is a necessary step towards investigating geometries over chain rings of larger nilpotency index, because of the nested structure of the projective Hjelmslev geometries. Finally, there exists a complete classification for the chain rings R with $|R| = q^2$, $R/\operatorname{rad} R \cong \mathbb{F}_q$.

In Section 2, we give some basic facts about finite chain rings and the structure of projective Hjelmslev geometries over such rings. In Section 3, we prove a necessary and sufficient condition for the existence of spresds in the projective Hjelmslev geometries $\mathrm{PHG}(R_R^{n+1})$, where R is a finite chain ring of nilpotency index 2. In Section 4, we present a construction for projective Hjelmslev palens from spreads in $\mathrm{PHG}(R_R^{n+1})$. Finally, some open problems are posed.

2. BASIC FACTS ON PROJECTIVE HJELMSLEV GEOMETRIES

A finite ring R (associative, with identity $1 \neq 0$, ring homomorphisms preserving the identity) is called a left (resp. right) chain ring if its lattice of left (resp. right) ideals forms a chain. It turns out that every left ideal is also a right ideal. Moreover, if $N = \operatorname{rad} R$ every proper ideal of R has the form $N^i = R\theta^i = \theta^i R$, for any $\theta \in N \setminus N^2$ and some positive integer i. The factors N^i/N^{i+1} are one-dimensional linear spaces over R/N. Hence, if $R/N \cong \mathbb{F}_q$ and m denotes the nilpotency index of N, the number of elements of R is equal to q^m. For further facts about chain rings, we refer to [3, 15, 16].

As mentioned above, we consider chain rings of nilpotency index 2, i.e. chain rings with $N \neq (0)$ and $N^2 = (0)$. Thus we have always $|R| = q^2$, where $R/N \cong \mathbb{F}_q$. Chain rings with this property have been classified in [4, 19]. If $q = p^r$ there are exactly $r+1$ isomorphism classes of such rings. These are:

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- for every $\sigma \in \mathrm{Aut}\,\mathbb{F}_q$ the ring $R_\sigma \cong \mathbb{F}_q[X;\sigma]/(X^2)$ of so-called σ-dual numbers over $\mathbb{F}_q$ with underlying set $\mathbb{F}_q \times \mathbb{F}_q$, component-wise addition and multiplication given by $(x_0,x_1)(y_0,y_1) = \big(x_0y_0, x_0y_1 + x_1\sigma(y_0)\big)$;
- the Galois ring $\mathrm{GR}(q^2,p^2) \cong \mathbb{Z}_{p^2}[X]/(f(X))$, where $f(X) \in \mathbb{Z}_{p^2}[X]$ is a monic polynomial of degree r, which is irreducible modulo p.

The rings R_σ with $\sigma \neq \mathrm{id}$ are noncommutative. Further R_{id} is commutative and $\mathrm{char}\,R_\sigma = p$ for every σ. The Galois ring $\mathrm{GR}(q^2,p^2)$ is commutative and has characteristic p^2. From now on we denote by R any finite chain ring of nilpotency index 2.

Let R be a finite chain ring and consider the module $M = R_R^k$. Denote by M^* the set of all non-torsion vectors of M, i.e. $M^* = M \setminus M\theta$. Define sets $\mathcal{P}$ and $\mathcal{L}$ by

$$\begin{aligned} \mathcal{P} &= \{xR; x \in M^*\}, \\ \mathcal{L} &= \{xR + yR; x,y \text{ linearly independent}\}, \end{aligned}$$

respectively, and take as incidence relation $I \subseteq \mathcal{P} \times \mathcal{L}$ set-theoretical inclusion. Further, define a neighbour relation $\asymp$ on the sets of points and lines of the incidence structure $(\mathcal{P},\mathcal{L},I)$ as follows:

(N1) the points $X,Y \in \mathcal{P}$ are neighbours (notation $X \asymp Y$) if there exist two different lines incident with both of them;
(N2) the lines $s,t \in \mathcal{L}$ are neighbours (notation $s \asymp t$) if there exist two different points incident with both of them.

The incidence structure $\Pi = (\mathcal{P},\mathcal{L},I)$ with the neighbour relation $\asymp$ is called the $(k-1)$-dimensional (right) projective Hjelmslev geometry over R and is denoted by $\mathrm{PHG}(R_R^k)$.

The point set $\mathcal{S} \subseteq \mathcal{P}$ is called a Hjelmslev subspace (or simply subspace) of $\mathrm{PHG}(R_R^k)$ if for every two points $X,Y \in \mathcal{S}$, there exists a line l incident with X and Y that is incident only with points of $\mathcal{S}$. The Hjelmslev subspaces of $\mathrm{PHG}(R_R^k)$ are of the form $\{xR; x \in (M')^*\}$, where M' is a free submodule of M. The (projective) dimension of a subspace is equal to the rank of the underlying module minus 1.

It is easily checked that $\asymp$ is an equivalence relation on each one of the sets $\mathcal{P}$ and $\mathcal{L}$. If $[X]$ denotes the set of all points that are neighbours to $X = xR$, then $[X]$ consists of all free rank 1 submodules of $xR + M\theta$. Similarly, the class $[l]$ of all lines which are neighbours to $l = xR + yR$ consists of all free rank 2 submodules of $xR + yR + M\theta$.

More generally, two subspaces $\mathcal{S}$ and $\mathcal{T}$, $\dim\mathcal{S} = s$, $\dim\mathcal{T} = t$, $s \leq t$, are neighbours if

$$\{[X]; X \in \mathcal{S}\} \subseteq \{[X]; X \in \mathcal{T}\}.$$

In particular, we say that the point X is a neighbour of the subspace $\mathcal{S}$ if there exists a point $Y \in \mathcal{S}$ with $X \asymp Y$. The neighbour class $[\mathcal{S}]$ contains all subspaces of dimension s that are neighbours to $\mathcal{S}$.

The next theorems give some insight into the structure of the projective Hjelmslev geometries $\mathrm{PHG}(R_R^k)$ and are part of more general results [1, 5, 8, 12, 13, 14, 21].

Theorem 2.1. *Let $\Pi = \mathrm{PHG}(R_R^k)$ where R is a chain ring with $|R| = q^2$, $R/N \cong \mathbb{F}_q$. Then*

(i) There are $q^{k-1} \cdot \frac{q^k-1}{q-1}$ points (hyperplanes) and $q^{2(k-2)} \cdot \frac{(q^k-1)(q^{k-1}-1)}{(q^2-1)(q-1)}$ lines in Π;
(ii) every point (hyperplane) has q^{k-1} neighbours;
(iii) every subspace of dimension $s-1$ is contained in exactly $q^{(t-s)(k-t)} \begin{bmatrix} k-s \\ t-s \end{bmatrix}_q$ subspaces of dimension $t-1$, where $s \leq t \leq k$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denotes the q-ary Gaussian binomial coefficient;
(iv) given a point P and a subspace $\mathcal{S}$ of dimension $s-1$ containing P, there exist exactly q^{s-1} points in $\mathcal{S}$ that are neighbours to P.

Note that the Hjelmslev spaces $\mathrm{PHG}(R_R^k)$ are 2-uniform in the sense of [5]. Denote by η the natural homomorphism from R^k to $R^k/R^k\theta$ and by $\overline{\eta}$ the mapping induced by η on the submodules of R^k. It is clear that for every point X and every line l we have

$$[X] = \{Y \in \mathcal{P}; \overline{\eta}(Y) = \overline{\eta}(X)\},$$

$$[l] = \{m \in \mathcal{L}; \overline{\eta}(m) = \overline{\eta}(l)\}.$$

Let us denote by $\mathcal{P}'$ (resp. $\mathcal{L}'$) the set of all neighbour classes of points (resp. lines). The following result is straightforward.

Theorem 2.2. *The incidence structure* $(\mathcal{P}', \mathcal{L}', I')$ *with incidence relation* I' *defined by*

$$[X]\, I'\, [l] \Longleftrightarrow \exists Y \in [X], \exists m \in [l] \colon Y\, I\, m$$

is isomorphic to the projective geometry $\mathrm{PG}(k-1,q)$

Let $\mathcal{S}_0$ be a fixed subspace in $\mathrm{PHG}(R_R^k)$ with $\dim \mathcal{S}_0 = s$. Define the set $\mathfrak{P}$ of subsets of $\mathcal{P}$ by

$$\mathfrak{P} = \{\mathcal{S} \cap [X]; X \asymp \mathcal{S}_0, \mathcal{S} \in [\mathcal{S}_0]\}.$$

The sets $\mathcal{S} \cap [X]$ are either disjoint or coincide. Define an incidence relation $\mathfrak{I} \subset \mathfrak{P} \times \mathcal{L}$ by

$$(\mathcal{S} \cap [X])\mathfrak{I}\, l \Longleftrightarrow l \cap (\mathcal{S} \cap [X]) \neq \emptyset.$$

Let $\mathcal{L}(\mathcal{S}_0)$ be the set of all lines in $\mathcal{L}$ incident with at least one point in $\mathfrak{P}$. For the lines $l_1, l_2 \in \mathcal{L}(\mathcal{S}_0)$ we write $l_1 \sim l_2$ if they are incident (under $\mathfrak{I}$) with the same elements of $\mathfrak{P}$. The relation $\sim$ is an equivalence relation under which $\mathcal{L}(\mathcal{S}_0)$ splits into classes of equivalent lines. Denote by $\mathfrak{L}$ a set of representatives of the equivalence classes of lines in $\mathcal{L}(\mathcal{S}_0)$. The set of representatives $\mathfrak{L}$ contains only two types of lines: lines l with $l \asymp \mathcal{S}_0$ and lines l with $l \not\asymp \mathcal{S}_0$.

Theorem 2.3. *The incidence structure* $(\mathfrak{P}, \mathfrak{L}, \mathfrak{I}\,|_{\mathfrak{P}\times\mathfrak{L}})$ *can be embedded into* $\mathrm{PG}(k-1,q)$.

A special case of this result is obtained if we take $\mathcal{S}_0$ to be a point. Given $\Pi = (\mathcal{P}, \mathcal{L}, I) = \mathrm{PHG}(R_R^k)$ and a point $P \in \mathcal{P}$, let $\mathcal{L}(P)$ be the set of all lines in $\mathcal{L}$ incident with points in $[P]$. For two lines $s, t \in \mathcal{L}(P)$ we write $s \sim t$ if s and t coincide on $[P]$. Denote by $\mathcal{L}_1$ a complete list of representatives of the lines from $\mathcal{L}(P)$ with respect to the equivalence relation $\sim$. Then we have the following result:

Theorem 2.4.

$$([P], \mathcal{L}_1, I|_{[P]\times\mathcal{L}_1}) \cong \mathrm{AG}(k-1,q).$$

Finally, let two points X_1 and X_2 in $\Pi = \mathrm{PHG}(R_R^k)$ be neighbours. Then any two lines incident with both X_1 and X_2 are neighbours and belong to the same class, $[l]$ say. In such case we say that the neighbour class $[l]$ has the direction of the pair (X_1, X_2).

3. Existence of spreads in projective Hjelmslev geometries

Definition 1. An r-spread of the projective Hjelmslev geometry $\mathrm{PHG}(R_R^{n+1})$ is a set $\mathcal{S}$ of r-dimensional subspaces such that every point is contained in exactly one subspace of $\mathcal{S}$.

Set

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\dots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\dots(q-1)}.$$

Theorem 3.1. *Let R be a chain ring with $|R| = q^2$, $R/\operatorname{rad} R \cong \mathbb{F}_q$. There exists a spread $\mathcal{S}$ of r-dimensional spaces of* $\mathrm{PHG}(R_R^n)$ *if and only if $r+1$ divides $n+1$.*

Proof. The number of points in an r-dimensional subspace is $q^r\begin{bmatrix} r+1 \\ 1\end{bmatrix}_q$. The existence of a spread of r dimensional subspaces implies that $q^r\begin{bmatrix} r+1 \\ 1\end{bmatrix}_q$ divides $q^n\begin{bmatrix} n+1 \\ 1\end{bmatrix}_q$, i.e. $\begin{bmatrix} r+1 \\ 1\end{bmatrix}_q$ divides $\begin{bmatrix} n+1 \\ 1\end{bmatrix}_q$, i.e. $r+1$ divides $n+1$.

Assume that $r+1$ divides $n+1$ and let s be determined by $n+1=(s+1)(r+1)$. First we consider the case where $R=\mathrm{GR}(q^2,p^2)$. Take an algebra of dimension $r+1$ over $R=\mathrm{GR}(q^2,p^2)$, e.g. let this algebra be $R_{r+1}=R[X]/(f(X))=\mathrm{GR}(q^{2(r+1)},p^2)$, where f is a monic irreducible polynomial of degree $r+1$ over R. If α is a root of f in R_{r+1} then every element β from R_{r+1} can be written as

$$\beta=b_0+b_1\alpha+\ldots+b_r\alpha^r,\ \ b_i\in R.$$

Clearly, R_{r+1}^{s+1}, R_{n+1} and R^{n+1} are isomorphic as modules over R. Thus each point in $\mathrm{PHG}(R_R^{n+1})$ can be represented by an $(s+1)$-tuple of elements from R_{r+1} or as a unit in R_{n+1}. In the same time, every $(s+1)$-tuple over R_{r+1} that has at least one coordinate that is a unit, can be viewed as a point in $\mathrm{PHG}(R_{r+1}^{s+1})$.

Let $(\gamma_0,\gamma_1,\ldots,\gamma_s)\in(R_{r+1}^{s+1})^*$ be a nontorsion vector. Without loss of generality, let $\gamma_0\neq 0$. Consider the system

$$\left|\begin{array}{lllllll} -\gamma_1x_0 & + & \gamma_0x_1 & & & & =0\\ -\gamma_2x_0 & + & & + & \gamma_0x_2 & & =0\\ \ldots & & & & \ddots & & =0\\ -\gamma_sx_0 & + & & & & +\ \ \gamma_0x_s & =0\end{array}\right. . \tag{1}$$

The choice of the nonzero element is not essential. If we take $\gamma_j\neq 0$. The system (1) is equivalent to $-\gamma_ix_j+\gamma_jx_i=0$ for $i=0,1,\ldots,s,\ i\neq j$. The solutions of (1) form a free submodule of rank 1 R_{r+1}^{s+1}, i.e. a point in $\mathrm{PHG}(R_{r+1}^{s+1})$. This rank 1 submodule can be considered as a free submodule of rank $(r+1)$ of R_R^{n+1}, i.e. a r-dimensional subspace of $\mathrm{PHG}(R_R^{n+1})$. Two $(r+1)$-dimensional subspaces in R_R^{n+1} obtained from different 1-dimensional subspaces of R_{r+1}^{s+1} do not have a common nontorsion vector.

Now consider two different points $(\gamma_0,\gamma_1,\ldots,\gamma_s)$ and $(\delta_0,\delta_1,\ldots,\delta_s)$ in $\mathrm{PHG}(R_{r+1}^{s+1})$. These points give rise to systems of the type (1) having as solutions different points of $\mathrm{PHG}(R_{r+1}^{s+1})$ (1-dimensional subspaces of R_{r+1}^{s+1}). Assume otherwise and let $(x_0,x_1,\ldots,x_s)\neq(0,0,\ldots,0)$ be a common solution of the two systems. Then

$$x_0=\lambda\gamma_0=\mu\delta_0,x_1=\lambda\gamma_1=\mu\delta_1,\ldots,x_s=\lambda\gamma_s=\mu\delta_s,$$

where $\lambda,\mu\in R_{r+1}$, $\lambda,\mu\neq 0$. This is a contradiction since the points $(\gamma_0,\gamma_1,\ldots,\gamma_s)$ and $(\delta_0,\delta_1,\ldots,\delta_s)$ were assumed to be different.

It remains to prove that every point is contained in a r-dimensional subspace. The number of points in $\mathrm{PHG}(R_R^{n+1})$ is $q^n\begin{bmatrix} n+1 \\ 1\end{bmatrix}_q$; the number of points in an r-dimensional subspace is $q^r\begin{bmatrix} r+1 \\ 1\end{bmatrix}_q$ and the number of points in $\mathrm{PHG}(R_{r+1}^{s+1})$ is $q^{s(r+1)}\begin{bmatrix} s+1 \\ 1\end{bmatrix}_{q^{r+1}}$. Now we have

$$q^r\begin{bmatrix} r+1 \\ 1\end{bmatrix}_q\cdot q^{s(r+1)}\begin{bmatrix} s+1 \\ 1\end{bmatrix}_{q^{r+1}}=q^r\frac{q^{r+1}-1}{q-1}\cdot q^{s(r+1)}\frac{q^{(s+1)(r+1)}-1}{q^{r+1}-1}=q^n\frac{q^{n+1}-1}{q-1}=q^n\begin{bmatrix} n+1 \\ 1\end{bmatrix}_q,$$

which means that the r-dimensional subspaces cover all points of $\mathrm{PHG}(R_R^{n+1})$.

Secondly, consider the case where R is the ring of σ dual numbers over the finite field $\mathbb{F}_q$, i.e. $R=R_\sigma=\mathbb{F}_q+t\mathbb{F}_q$. Denote by R' the ring of σ'-dual numbers $\mathbb{F}_{q^{r+1}}+t\mathbb{F}_{q^{r+1}}$, where $\sigma'|_{\mathbb{F}_{q^{r+1}}}=\sigma$. Similarly, let R'' be the ring of σ''-dual numbers $\mathbb{F}_{q^{n+1}}+t\mathbb{F}_{q^{n+1}}$, where $\sigma''|_{\mathbb{F}_{q^{n+1}}}=\sigma'$. The ring R is a subring of R' which in turn is a subring of R''. As above, $R_R'^{s+1}$ and R_R^{n+1} and R_R'' are isomorphic as (right) submodules over R.

Consider an arbitrary nontorsion vector $(\gamma_0,\gamma_1,\ldots,\gamma_s)\in {}_RR'^{s+1}$. Fix a component which is a unit, γ_0 say, and consider the system of linear equations (1). The set of solutions of (1) is a free rank 1 submodule of R'^{s+1}_R which can be viewed as a free rank r submodule of R^{n+1}_R. Further the proof is completed as for Galois rings. □

Remark 3.1. Assume $r+1$ divides $n+1$. We can prove the existence of a spread of r-dimensional subspaces using the nested structure of the projective Hjelmslev geometries.

Let $\mathcal{H}_0$ be a fixed subspace in $\mathrm{PHG}(R^k_R)$ with $\dim \mathcal{H}_0 = r$. Define the set $\mathfrak{P}$ of subsets of $\mathcal{P}$ by

$$\mathfrak{P} = \{\mathcal{H} \cap [X]; X \asymp \mathcal{H}_0, \mathcal{H} \text{ is a subspace}, \dim \mathcal{H} = s, \mathcal{H} \in [\mathcal{H}_0]\}.$$

The sets $\mathcal{H} \cap [X]$ are either disjoint or coincide. Define an incidence relation $\mathfrak{I} \subset \mathfrak{P} \times \mathcal{L}$ by

$$(\mathcal{H} \cap [X])\mathfrak{I}\, l \Longleftrightarrow l \cap (\mathcal{H} \cap [X]) \neq \emptyset.$$

Let $\mathcal{L}(\mathcal{H}_0)$ be the set of all lines in $\mathcal{L}$ incident with at least one point in $\mathfrak{P}$. For the lines $l_1, l_2 \in \mathcal{L}(\mathcal{H}_0)$ we write $l_1 \sim l_2$ if they are incident (under $\mathfrak{I}$) with the same elements of $\mathfrak{P}$. The relation $\sim$ is an equivalence relation under which $\mathcal{L}(\mathcal{H}_0)$ splits into nonintersecting classes of equivalent lines. Denote by $\mathfrak{L}$ a set of representatives of the equivalence classes of lines in $\mathcal{L}(\mathcal{H}_0)$. The set of representatives $\mathfrak{L}$ contains only two types of lines: lines l with $l \asymp \mathcal{H}_0$ and lines l with $l \not\asymp \mathcal{H}_0$.

It is known from [8] that the incidence structure $(\mathfrak{P}, \mathfrak{L}, \mathfrak{I}\,|_{\mathfrak{P}\times\mathfrak{L}})$ can be embedded isomorphically into the projective plane $\mathrm{PG}(k-1,q)$. Hence we can construct a spread in $\mathrm{PHG}(R^{n+1}_R)$ in the following way. We start with a spread in the factor geometry $PG(n,q)$. This spread defines a set of neighbourhood classes of projective r-subspaces. Each one of these classes is isomorphic in the sense of the above mentioned result to a projective geometry $\mathrm{PG}(n,q)$ with an $(n-r-1)$-dimensional space deleted. Now it suffices to take a spread which contains a spread of the deleted $(n-r-1)$-dimensional subspace.

A spread with this property can be constructed, for instance, by repeating the construction from the proof of Theorem 3.1. The exceptional $(n-r-1)$-dimensional space can be taken as the space consisting of all points having zeros in the first $r+1$ positions.

Our result can be generalized to projective Hjelmslev geometries over arbitrary chain rings R.

Theorem 3.2. *Let R be a chain ring with $|R| = q^m$, $R/\operatorname{rad} R \cong \mathbb{F}_q$. The n-dimensional projective Hjelmslev geometry* $\mathrm{PHG}(R^{n+1}_R)$ *has a spread of r-dimensional projective Hjelmslev subspaces if and only if $r+1$ divides $n+1$.*

Proof. We give only a sketch of a proof. For the sake of convenience, we set $N = \operatorname{rad} R = \theta R$. As before, the "only if"-part is straightforward. The proof of the "if"-part uses induction by m and n. So far, we have proved this result for $m = 1$ and 2. It is also trivial for $n = r$ for every m.

Consider the factor geometry having as points the $(m-1)$-neighbour classes on points. It is isomorphic to $\mathrm{PHG}((R^k/\theta^{m-i}R^k)_{R/N^{m-i}}$ (cf. [8]). By the induction hypothesis, it has a spread of r-dimensional projective Hjelmslev subspaces. The preimage of these subspaces are of the form $[\Delta]_{m-1}$ where Δ is an r-dimensional Hjelmslev subspace in $\mathrm{PHG}(R^{n+1}_R)$. Here $[\Delta]_j$ is the class of all r-dimensional Hjelmslev subspaces that are j-th neighbours to Δ. Now $[\Delta]_j$ can be imbedded isomorphically in $\mathrm{PHG}((R/N)^{n+1}_{R/N}) \cong \mathrm{PG}(n,q)$ (cf. [8]) where the missing part is an $(n-r-1)$-dimensional subspace, $\mathcal{H}$ say. Since $r+1$ divides $n-r = (n+1)-(r+1)$, we have that $\mathcal{H}$ contains a spread by the induction hypothesis. Now it is enough to take a spread which contains as a subset the spread of the missing $(n-r-1)$-dimensional subspace. □

4. PROJECTIVE HJELMSLEV PLANES FROM SPREADS

Spreads in $\Pi = \mathrm{PHG}(R^{n+1}_R)$ can be used to construct projective Hjelmslev planes. Set

$$n = 2t-1,\ r = t-1,\ s = 1.$$

By Theorem 3.1 , there exists a spread $\mathcal{S}$ of r-dimensional subspaces of Π such that its image under the canonical map $\pi = \pi^{(1)}$ is a (multiple of a) spread in $\mathrm{PG}(n,q)$.

The geometry Π can be imbedded in $\widehat{\Pi} = \mathrm{PHG}(R_R^{n+2})$, e.g. by taking by taking as points of Π all points of $\widehat{\Pi}$ with first coordinate 0. Hence Π can be considered as a hyperplane of $\widehat{\Pi}$. Denote by $[\Pi]$ the set of all neighbour hyperpalnes to Π in $\widehat{\Pi}$. Define a new incidence structure as follows:

Take as points:

(1) all points of $\widehat{\Pi}$ that are not incident with a point of $[\Pi]$. These are called proper points and their number is:
$$q^{n+1}\frac{q^{n+2}-1}{q-1} - q^{n+1}\frac{q^{n+1}-1}{q-1} = q^{2(n+1)} = q^{4t}.$$

(2) all subspaces of the form
$$\langle S,P\rangle \cap H,$$
where S is an r-dimensional subspace from $\mathcal{S}$, P is a point from $\widehat{\Pi}\setminus[\Pi]$ and H is a hyperplane of $\widehat{\Pi}$ contained in the neighbour class $[\Pi]$. These are called ideal points. The number of choices for the point P is $q^{4t} = q^2(n+1)$. The number of choices for $S \in \mathcal{S}$ is
$$|\mathcal{S}| = \frac{q^n\frac{q^{n+1}-1}{q-1}}{q^r\frac{q^{r+1}-1}{q-1}} = \frac{q^{2t-1}(q^{2t}-1)}{q^{t-1}(q^t-1)} = q^t(q^t+1).$$
The number of choices for $H \in [\Pi]$ is $q^{n+1} = q^{2t}$. For all points Q in $\langle S,P\rangle\setminus[\Pi]$, we have $\langle S,Q\rangle \cap H = \langle S,P\rangle \cap H$ i.e. we get the same point in the new incidence structure. Hence for
$$q^{r+1}\frac{q^{r+2}-1}{q-1} - q^{r+1}\frac{q^{r+1}-1}{q-1} = q^{2(r+1)} = q^{2t}.$$
different points P we get the same $(r+1)$-dimensional subspace $\langle S,P\rangle$.

As lines we take:

(1) all subspaces of the form $\langle S,P\rangle$, where $S \in \mathcal{S}$ and P is a point from $\widehat{\Pi}\setminus[\Pi]$, i.e. these are all $(r+1)$-dimensional subspaces through r-dimensional subspaces in the spread;
(2) all hyperplanes H from $[\Pi]$.

For the proper points neighbourhood is inherited from $\widehat{\Pi}$. For the ideal points, we have that
$$\langle S',P\rangle \cap H' \asymp \langle S'',P\rangle \cap H''$$
if and only if S' and S'' are neighbours in Π. By definition, two lines ℓ_1 and ℓ_2 are neighbours if for every point $X \in \ell_1$ there exists a point $Y \in \ell_2$ with $X \asymp Y$, and, conversely, for every $Y \in \ell_2$ there exists an $X \in \ell_1$ with $Y \asymp X$.

Lemma 4.1. *Let S be an r-dimensional subspace in Π and let P,Q be points from $\widehat{\Pi}\setminus[\Pi]$ with $P \asymp Q$. Then $\langle S,P\rangle \cap [\Pi] = \langle S,Q\rangle \cap [\Pi]$.*

Proof. Assume there exists a point $X \in [\Pi]$ with $X \in \langle S,Q\rangle$, but $X \notin \langle S,P\rangle$. The lines PY and QY are neighbours. Therefore $|PY \cap QY| = q$. The common points of both lines must be neighbours to Y. Hence the q common points must lie in $[\Pi]$., contradiction to the initial assumption. □

Lemma 4.2. *The number of hyperplanes from $[\Pi]$ through a fixed r-dimensional flat $S \in \mathcal{S}$ ($S \subset \Pi$) is q^t.*

Proof. Any r-dimensional flat in an $(n+1)$-dimensional space can be given by a set of $(n+1)-r = t+1$ equations. Without loss of generality, let S be given by $x_0 = x_1 = \ldots = x_t = 0$ and let Π be the hyperplane defined by $x_0 = 0$. An arbitrary hyperplane in $[\Pi]$ containing S satisfies an equation of the form:
$$x_0 + \theta(r_1x_1 + r_2x_2 + \ldots + r_tx_t) = 0. \qquad (2)$$
We have $\theta r = \theta s$ if and only if $r - s \in \mathrm{rad}\,R$, therefore (2) describes all hyperplanes in $[\Pi]$ through S when $(r_1, r_2, \ldots, r_t)$ runs Γ^t, where Γ is a set of elements no two of which are congruent modulo $\mathrm{rad}\,R$. hence there are exactly q possibilities for each r_i and the number of hyperplanes in $[\Pi]$ through S is q^t. □

According to Lemma 4.1 the number of the essentially different choices of P is

$$\frac{\frac{q^{n+2}-1}{q-1}-\frac{q^{n+1}-1}{q-1}}{\frac{q^{r+2}-1}{q-1}-\frac{q^{r+1}-1}{q-1}}=\frac{q^{n+1}}{q^{r+1}}=q^t.$$

The number of choices for S is $q^t(q^t+1)$ and the number of hyperplanes H from $[\Pi]$ is q^{2t}. On the other hand, by Lemma 4.2, we get the same intersection for q^t different hyperplanes in $[\Pi]$. Each ideal point of the second type can be obtained for q^t different subspaces $\langle S,P\rangle$. therefore the number of all points of the second type is

$$\frac{q^t\cdot q^t(q^t+1)\cdot q^{2t}}{q^t\cdot q^t}=q^{3t}+q^{2t}.$$

Now it is a straightforward check that the defined incidence structure is indeed a projective Hjelmslev plane.

5. Open problems

(1) Is it possible to exist a spread whose image under the cannonical map π is not a (multiple of a) spread in $\mathrm{PG}(k-1,q)$?
(2) Taking different spreads in $\mathrm{PHG}(R_R^{n+1})$ we can construct nonisomorphic projective Hjelmslev planes. Under what conditions do we obtain non-Desarguesian planes?
(3) In which case do we obtain coordinate planes? Can we determine the underlying ring from the spread?

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