New nonexistence results for spherical 5-designs

SILVIA BOUMOVA, PETER BOYVALENKOV[1] silvi,peter@moi.math.bas.bg
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences,
8 G.Bonchev str., 1113, Sofia, BULGARIA

HRISTINA KULINA kulina@pu.acad.bg
Faculty of Mathematics and Informatics, Plovdiv University,
236 Bulgaria Blvd., 4003 Plovdiv, BULGARIA

MAYA STOYANOVA stoyanova@fmi.uni-sofia.bg
Faculty of Mathematics and Informatics, Sofia University,
5 James Baucher blvd, 1164 Sofia, BULGARIA

Abstract. We investigate the structure of spherical 5-designs of relatively small cardinalities. We obtain some bounds on the extreme inner products of such designs. As a result, in 42 cases we prove nonexistence of designs of corresponding parameters. Our approach can be applied for other strengths and cardinalities.

1 Introduction

The spherical designs were introduced in 1977 by Delsarte-Goethals-Seidel [8].

Definition 1. *A spherical τ-design $C \subset \mathbb{S}^{n-1}$ is a finite nonempty subset of $\mathbb{S}^{n-1}$ such that*

$$\frac{1}{\mu(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} f(x) d\mu(x) = \frac{1}{|C|} \sum_{x \in C} f(x) \qquad (1)$$

($\mu(x)$ is the Lebesgue measure) holds for all polynomials $f(x) = f(x_1, x_2, \ldots, x_n)$ of degree at most τ (i.e. the average of $f(x)$ over the set C is equal to the average of $f(x)$ over $\mathbb{S}^{n-1}$). The number $\tau = \tau(C)$ is called strength of C.

Denote by $B(n,\tau)$ the minimum possible cardinality of a τ-design on $\mathbf{S}^{n-1}$, i.e.

$$B(n,\tau) = \min\{|C| : C \in \mathbf{S}^{n-1} \text{ is a } \tau\text{-design}\}.$$

Delsarte-Goethals-Seidel [8] prove the following lower bound for $B(n,\tau)$, i.e.

$$B(n,\tau) \geq D(n,\tau) = \begin{cases} 2\binom{n+e-2}{n-1}, & \text{if } \tau = 2e-1, \\ \\ \binom{n+e-1}{n-1} + \binom{n+e-2}{n-1}, & \text{if } \tau = 2e. \end{cases}$$

[1]The author is also with South-Western University, Blagoevgrad, Bulgaria.

In this paper we prove nonexistence of certain spherical 5-designs. This does not give direct improvement to the above bound for $\tau = 5$ but sheds some light on the problem for existence of designs of prescribed dimension, strength and cardinality.

The following equivalent definition of spherical designs is very suitable for our purposes.

Definition 2. *A spherical τ-design $C \subset \mathbb{S}^{n-1}$ is a finite nonempty subset of $\mathbb{S}^{n-1}$ such that for any point $x \in C$ and any real polynomial $f(t)$ of degree at most τ, the equality*

$$\sum_{y \in C \setminus \{x\}} f(\langle x, y \rangle) = f_0 |C| - f(1) \tag{2}$$

holds, where f_0 is the first coefficient in the expansion of $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$ in terms of the Gegenbauer polynomials [1, Chapter 22].

We are interested in the following:

Problem. Given dimension n and cardinality M decide whether a 5-design on $\mathbb{S}^{n-1}$ of cardinality $|C| = M$ exists.

We obtain restrictions on the structure of 5-designs of relatively small cardinalities, i.e. close to $D(n, 5) = n(n + 1)$. This allows us to obtain some nonexistence results. Our method can be applied for other odd strengths and cardinalities.

All known constructions of spherical designs (see, for example, [2, 3, 9, 11]) suggest that the structure of the design with respect to any of its points should be investigated. This can be done by using suitable polynomials in (1) combined with some geometric arguments.

In Section 2 we describe our approach. The results are formulated in general but will be used for $\tau = 5$. In fact, we continue investigations started in [6, 5, 4] with proving nonexistence of designs in many cases. The results for $\tau = 5$ in dimensions $n \leq 25$ are presented in sections 3 and 4.

It was proved in [5] that the condition $\rho_0|C| \geq 2$ is necessary for the existence of τ-designs $C \subset \mathbb{S}^{n-1}$ with odd τ and $|C|$. For 5-designs, we prove (ruling out 42 cases) that in dimensions $5 \leq n \leq 25$ this can be replaced by $\rho_0|C| > 3$.

2 Preliminaries

Let $C \in \mathbb{S}^{n-1}$ be a spherical τ-design, where $\tau = 2e - 1 \geq 3$ is odd. For every point $x \in C$ we consider the inner products of x with all other points of C, i.e.

$$I(x) = \{\langle u, x \rangle : u \in C \setminus \{x\}\} = \{t_1(x), t_2(x), \ldots, t_{|C|}(x)\},$$

where $-1 \leq t_1(x) \leq t_2(x) \leq \cdots \leq t_{|C|-1}(x) < 1$. Using suitable polynomials in (1) we obtain lower and upper bounds for the extreme inner products in $I(x)$ for some special points x. Let us recall some results from [10, 5, 4].

It follows from [10, Section 4] (see also [5]) that for every fixed cardinality $|C| \geq D(n, 2e-1)$ there exist uniquely determined real numbers $-1 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_{e-1} < 1$ and $\rho_0, \rho_1, \ldots, \rho_{e-1}$, $\rho_i > 0$ for $i = 0, 1, \ldots, e-1$, such that the equality

$$f_0 = \frac{f(1)}{|C|} + \sum_{i=0}^{e-1} \rho_i f(\alpha_i) \tag{3}$$

is true for every real polynomial $f(t)$ of degree at most $2e-1$. We use (3) in some calculations of $f_0|C| - f(1)$ in the right hand side of (2). Another useful formula for f_0 is

$$f_0 \;\;=\;\; a_0 + \sum_{i=1}^{[k/2]} \frac{a_{2i}(2i-1)!!}{n(n+2)\cdots(n+2i-2)} = a_0 + \frac{a_2}{n} + \frac{3a_4}{n(n+2)} + \cdots , \tag{4}$$

where $f(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k$.

The numbers α_i, $i = 0, 1, \ldots, e-1$ are all roots of the equation

$$P_e(t)P_{e-1}(s) - P_e(s)P_{e-1}(t) = 0,$$

where $P_i(t) = P_i^{(n-1)/2,(n-3)/2}(t)$ is a Jacobi polynomial [1]. The weights ρ_i can be calculated by

$$\rho_i = -\frac{\prod_{0 \leq j \leq e-1, j \neq i}(1-\alpha_j^2)}{\alpha_i |C| \prod_{0 \leq j \leq e-1, j \neq i}(\alpha_i^2 - \alpha_j^2)}.$$

Theorem 1. [5] *If $C \subset \mathbb{S}^{n-1}$ is a τ-design with odd $\tau = 2e-1$ and odd $|C|$ then $\rho_0|C| \geq 2$.*

Lemma 1. [5] *Let $C \subset \mathbb{S}^{n-1}$ be a τ-design with odd $\tau = 2e-1$. For any point $x \in C$ we have $t_1(x) \leq \alpha_0$ and $t_{|C|-1}(x) \geq \alpha_{e-1}$. If $|C|$ is odd then there exist a point $x \in C$ such that $t_2(x) \leq \alpha_0$.*

Lemma 2. [4] *Let $C \subset \mathbb{S}^{n-1}$ be a τ-design with odd $\tau = 2e-1$ and of odd cardinality $|C|$. Then there exist three distinct points $x, y, z \in C$ such that $t_1(x) = t_1(y)$ and $t_2(x) = t_1(z)$. Moreover, we have $t_{|C|-1}(z) \geq \max\{\alpha_{e-1}, 2\alpha_0^2 - 1\}$.*

It is convenient to use the following notation: $U_{\tau,i}(x)$ (respectively $L_{\tau,i}(x)$) for any upper (resp. lower) bound on the inner product $t_i(x)$. When a bound does not depend on x we omit x in the notation. For example, the first bound from Lemma 1 is $t_1(x) \leq U_{\tau,1} = \alpha_0$ and the last bound from Lemma 2 is $t_{|C|-1}(z) \geq L_{\tau,|C|-1}(z) = \max\{\alpha_{e-1}, 2\alpha_0^2 - 1\}$.

3 General bounds

In what follows we take $\tau = 5$. Let $C \subset \mathbb{S}^{n-1}$ be a 5-design of odd cardinality $|C|$ and $x, y, z \in C$ be points as in Lemma 2. Then α_0, α_1 and α_2 are the roots of the equation

$$n(1-\alpha)((n+2)(n+3)\alpha^2 + 4(n+2)\alpha - n + 1) = 2\alpha|C|(3 - (n+2)\alpha^2).$$

We denote $g(t) = (t-\alpha_1)^2(t-\alpha_2)^2$.

After [4], there are 42 open cases where $5 \leq n \leq 25$, $|C|$ is odd and $2 \leq \rho_0|C| \leq 3$. In all these cases $2\alpha_0^2 - 1 > \alpha_2$ and Lemma 2 gives

$$t_{|C|-1}(z) \geq L_{5,|C|-1}(z) = 2\alpha_0^2 - 1. \tag{5}$$

We focus on the inner products in $I(x)$ and $I(z)$. The main purpose is obtaining a upper bound

$$t_1(z) \leq U_{5,1}(z) < \alpha_0. \tag{6}$$

We start with a lower bound on $t_1(z) = t_2(x)$.

Lemma 3. *We have $t_1(z) \geq L_{5,1}(z)$ where $L_{5,1}(z)$ is the smallest root of the equation $2g(t) = \rho_0|C|g(\alpha_0)$.*

Proof. Use (2) with $g(t)$ for x and C and (3) for the left hand side. For $g(t)$ we have $g(t_i(x)) \geq 0$ for $i \geq 3$, $g(t_1(x)) \geq g(t_2(x))$, because $g(t)$ is decreasing in $(-\infty, \alpha_1)$ and $t_1(x) \leq t_2(x) \leq \alpha_0$. Therefore

$$\begin{aligned} \rho_0 g(\alpha_0)|C| &= g_0|C| - g(1) = \sum_{i=1}^{|C|-1} g(t_i(x)) \\ &\geq g(t_1(x)) + g(t_2(x)) \geq 2g(t_2(x)) = 2g(t_1(z)), \end{aligned}$$

since $t_1(z) = t_2(x)$ by Lemma 2. Now the conclusion follows by using again that $g(t)$ is decreasing in $(-\infty, \alpha_1)$. $\square$

We illustrate our method with two examples which appear in parts after each important assertion. The numerical results are approximated as follows: the lower bounds are rounded up and the upper bounds are truncated as we usually give six digits after the decimal point. We do the same in all numerical applications. The calculations were performed by a MAPLE programme. Both the programme and the calculations for every separate case can be obtained from the authors upon request.

In all appearances of Example 1 (resp. Example 2) we treat the case $n = 11$ and $|C| = 147$ (resp. the case $n = 17$ and $|C| = 343$). Both examples contain the complete nonexistence proofs for the corresponding designs.

Example 1. For $n = 11$ and $|C| = 147$, we have $\alpha_0 = -0.830399$, $\alpha_1 = -0.248366$ and $\alpha_2 = 0.293051$. Then the equation from Lemma 3 is approximated by $(t + 0.248366)^2(t - 0.293051)^2 = 0.582577$ whose smallest root is approximately -0.892289. Therefore $t_1(z) \geq L_{5,1}(z) = -0.892289$. $\diamond$

Example 2. Analogously, for $n = 17$ and $|C| = 343$, we approximate $\alpha_0 = -0.816081$, $\alpha_1 = -0.210608$, $\alpha_2 = 0.240641$ and $t_1(z) \geq L_{5,1}(z) = -0.892617$. $\diamond$

Lemma 3 allows us to obtain a good upper bound on $t_2(z)$.

Lemma 4. *We have $t_2(z) \leq U_{5,2}(z)$ with $U_{5,2}(z)$ defined in the proof.*

Proof. We denote $q(t) = t^2 + at + b$ and use (2) with $f(t) = (t - t_2(z))q^2(t)$ for z and C where the parameters a and b will be determined later but have to be such that the polynomial $q(t)$ has two real roots in $[\alpha_0, \alpha_2]$. We have $f(t_i(z)) \geq 0$ for $i \geq 2$, $f(t_1(z)) \geq f(L_{5,1}(z))$ because $f(t)$ is increasing in $(-\infty, t_2(z))$. Therefore

$$\begin{aligned} f_0|C| - f(1) &= \sum_{i=1}^{|C|-1} f(t_i(z)) \\ &\geq f(t_1(z)) + f(t_{|C|-1}(z)) \geq f(L_{5,1}(z)) + f(L_{5,|C|-1}(z)) \end{aligned}$$

(we use (5) and $t_1(z) \geq L_{5,1}(z)$ by Lemma 3). This gives the following inequality for $t_2(z)$

$$t_2(z) \leq F(a,b) = \frac{A(a,b)}{B(a,b)},$$

where

$$A(a,b) = \frac{6a|C|}{n(n+2)} + \frac{2ab|C|}{n} - q^2(1) - L_{5,1}(z)q^2(L_{5,1}(z)) - L_{5,|C|-1}(z)q^2(L_{5,|C|-1}(z))$$

and

$$B(a,b) = \frac{3|C|}{n(n+2)} + \frac{(a^2+b)|C|}{n} + b^2|C| - q^2(1) - q^2(L_{5,1}(z)) - q^2(L_{5,|C|-1}(z)).$$

After the optimization over a and b we obtain the bound $t_2(z) \leq U_{5,2}(z)$. $\square$

Example 1. (Continued) We have $t_2(z) \leq U_{5,2}(z) = -0.774411$. $\diamond$

Example 2. (Continued) We have $t_2(z) \leq U_{5,2}(z) = -0.744010$. $\diamond$

Lemma 5. *We have*

$$t_{|C|-1}(x) \geq L_{5,|C|-1}(x) = U_{5,1}(z)U_{5,2}(z) - \sqrt{(1 - U_{5,1}^2(z))(1 - U_{5,2}^2(z))}, \tag{7}$$

where $U_{5,i}(z)$, $i = 1, 2$, is any (good) upper bound for $t_i(z)$, $i = 1, 2$.

Proof. Denote by φ and ψ the acute angles such that $\cos\varphi = -U_{5,1}(z)$ and $\cos\psi = -U_{5,2}(z)$. Let $u \in C$ be such that $\langle u, z\rangle = t_2(z)$. Then the angle between the vectors x and u is at most $\varphi + \psi$ and we have

$$\begin{aligned} t_{|C|-1}(x) &\geq \langle x, u\rangle \geq \cos(\varphi + \psi) \\ &= U_{5,1}(z)U_{5,2}(z) - \sqrt{(1 - U_{5,1}^2(z))(1 - U_{5,2}^2(z))}. \qquad \square \end{aligned}$$

Lemma 6. *We have $t_3(z) \geq \min\{L_{5,3}(z), \alpha_1\}$, where $L_{5,3}(z)$ is the smallest root of the equation $2g(t) = (\rho_0|C| - 1)g(\alpha_0) - g(L_{5,|C|-1}(z))$.*

Proof. Use (2) with $g(t)$ for z and C. Applying similar arguments as in Lemma 1 and assuming $t_3(z) < \alpha_1$, we consecutively obtain

$$\begin{aligned} \rho_0 g(\alpha_0)|C| &= g_0|C| - g(1) = \sum_{i=1}^{|C|-1} g(t_i(z)) \\ &\geq g(t_1(z)) + g(t_2(z)) + g(t_3(z)) + g(t_{|C|-1}(z)) \\ &\geq g(\alpha_0) + 2g(t_3(z)) + g(L_{5,|C|-1}(z)). \end{aligned}$$

This implies the assertion since $g(t)$ is decreasing in $(-\infty, \alpha_1)$. $\square$

Example 1. (Continued) We have $t_3(z) \geq L_{5,3}(z) = -0.801894$. $\diamond$

Example 2. (Continued) We have $t_3(z) \geq L_{5,3}(z) = -0.807530$. $\diamond$

In all 42 cases under consideration we have the following ordering for our parameters and the bounds from Lemmas 1-6:

$$-1 < L_{5,1}(z) < \alpha_0 < L_{5,3}(z) < U_{5,2}(z) < \alpha' < \alpha_1 < \alpha_2 < L_{5,|C|-1}(z),$$

where α' is the smallest root of the derivative $f'(t)$ of the polynomial $f(t) = (t - \alpha_0)g(t)$.

We further consider several cases for the location of the inner products $t_2(z)$ and $t_3(z)$. The details will be shown in the next section.

4 The location of $t_2(z)$ and $t_3(z)$

Using the bounds from Lemmas 1-6 we consider two cases for the location of $t_2(z)$ with respect to α_0.

4.1 Case 1: $t_2(z) \in [\alpha_0, U_{5,2}(z)]$

We are ready to obtain better upper bound on $t_1(z)$ as required by (6).

Lemma 7. *We have $t_1(z) \leq U_{5,1}(z)$, where $U_{5,1}(z)$ is the smallest root of the equation $f(t) = (\rho_0|C|-1)f(\alpha_0) - f(L_{5,|C|-1}(z))$ and $f(t) = (t - L_{5,3}(z))g(t)$.*

Proof. Use (2) with $f(t)$ for z and C. We have

$$\begin{aligned}\rho_0|C|f(\alpha_0) &= f_0|C| - f(1) = \sum_{i=1}^{|C|-1} f(t_i(z)) \\ &\geq f(t_1(z)) + f(t_2(z)) + f(t_{|C|-1}(z)) \\ &\geq f(t_1(z)) + f(\alpha_0) + f(L_{5,|C|-1}(z)),\end{aligned}$$

which implies the inequality $t_1(z) \leq U_{5,1}(z)$ since $f(t)$ is increasing in $(-\infty, \alpha_0)$. $\square$

Remark. According to (6) we need $U_{5,1}(z) < \alpha_0$. This is the case when

$$\rho_0|C| < 2 + \frac{f(2\alpha_0^2 - 1)}{f(\alpha_0)}.$$

Example 1. (Continued) We have $t_1(z) \leq U_{5,1}(z) = -0.852885$. $\diamond$

Example 2. (Continued) We have $t_1(z) \leq U_{5,1}(z) = -0.829252$. $\diamond$

Having a good upper bound $t_1(z) \leq U_{5,1}(z)$ we are in a position to obtain strong necessary condition for the existence of C. We use $t_2(x) = t_1(z) \leq U_{5,1}(z)$ and $t_{|C|-1}(x) \geq L_{5,|C|-1}(x)$ by Lemmas 2 and 5 respectively.

The next Lemma gives a necessary condition for the existence of C. It uses the information about $I(x)$ which is collected so far. We denote shortly this check for existence by check(x).

Lemma 8. (Check for existence by x) *If there exist $a, b \in [\alpha_0, \alpha_2]$ such that*

$$\text{check}(x) := h_0|C| - h(1) - 2h(U_{5,1}(z)) - h(L_{5,|C|-1}(x)) < 0,$$

where $h(t) = (t-a)^2(t-b)^2$, then C does not exist.

Proof. Use (2) with $h(t)$, x and C. We have

$$\begin{aligned}h_0|C| - h(1) &= \sum_{i=1}^{|C|-1} h(t_i(x)) \geq h(t_1(x)) + h(t_2(x)) + h(t_{|C|-1}(x)) \\ &\geq 2h(t_2(x)) + h(L_{5,|C|-1}(x)) \geq 2h(U_{5,1}(z)) + h(L_{5,|C|-1}(x))\end{aligned}$$

(we use $t_1(x) \leq t_2(x) = t_1(z) \leq U_{5,1}(z)$ and $t_{|C|-1}(x) \geq L_{5,|C|-1}(x)$), which implies the assertion since $h(t)$ is decreasing in $(-\infty, \alpha_0)$ and increasing in $(\alpha_2, +\infty)$. $\square$

Example 3. For $n = 12$, $|C| = 171$ we have $t_1(z) \leq U_{5,1}(z) = -0.887772$, $L_{5,|C|-1}(x) = 0.501028$ and $\text{check}(x) = -0.038759 < 0$. Therefore 5-designs on $\mathbb{S}^{11}$ with 171 points such that $t_2(z) \in [\alpha_0, U_{5,2}(z)]$ do not exist. In fact this case was ruled out by Boumova-Boyvalenkov-Danev in [4]. $\diamond$

After the optimization of a and b we can still have $\text{check}(x) \geq 0$ (converse to the inequality from Lemma 8). Then we continue with a recursive procedure which replaces α_0 with $U_{5,1}(z)$ whenever possible in Lemma 7 and again turn to Lemma 8 with better $U_{5,1}(z)$ and $L_{5,|C|-1}(x)$.

Example 1. (Continued) At the first step we have $t_1(z) \leq U_{5,1}(z) = -0.852885$, $L_{5,|C|-1}(x) = 0.330162$ and $\text{check}(x) = 0.195233 > 0$. The second step gives $t_1(z) \leq U_{5,1}(z) = -0.872394$, $L_{5,|C|-1}(x) = 0.366334$ and $\text{check}(x) = 0.099869 > 0$ again, but the third step gives $t_1(z) \leq U_{5,1}(z) = -0.903327$, $L_{5,|C|-1}(x) = 0.428156$ and $\text{check}(x) = -0.072318 < 0$. Therefore Lemma 8 implies that 5-designs on $\mathbb{S}^{10}$ with 147 points such that $t_2(z) \in [\alpha_0, U_{5,2}(z)]$ do not exist. $\diamond$

Example 2. (Continued) Similarly, after six steps we obtain $t_1(z) \leq U_{5,1}(z) = -0.926518$, $L_{5,|C|-1}(x) = 0.437940$ and $\text{check}(x) = -0.218084 < 0$. Therefore 5-designs on $\mathbb{S}^{16}$ with 343 points such that $t_2(z) \in [\alpha_0, U_{5,2}(z)]$ do not exist. $\diamond$

This approach rules out 40 cases out of the 42 under consideration. The two remaining cases ($n = 7$, $|C| = 63$ and $n = 8$, $|C| = 81$) are ruled out by precise consideration how close is $t_3(z)$ to $L_{5,3}(z)$.

We have $t_2(z) \in [\alpha_0, U_{5,2}(z)]$ and consider two possibilities for $t_3(z)$.

Case 1.1. Let us have $t_3(z) \in [L_{5,3}(z), L_{5,3}(z) + \varepsilon]$, where $\varepsilon > 0$ is a positive number such that $L_{5,3}(z) + \varepsilon < \alpha_1$. We have the analog of Lemma 8 (check for existence which uses the information about $I(z)$).

Lemma 8′. (Check for existence by z) *If there exist $a, b \in [\alpha_1, \alpha_2]$ such that*

$$\text{check}(z) := h_0|C| - h(1) - h(U_{5,1}(z)) - 2h(L_{5,3}(z) + \varepsilon) - h(L_{5,|C|-1}(z)) < 0,$$

where $h(t) = (t-a)^2(t-b)^2$, then C does not exist.

Proof. Use (2) with $h(t)$, z and C. We have

$$\begin{aligned} h_0|C| - h(1) &= \sum_{i=1}^{|C|-1} h(t_i(z)) \geq h(t_1(z)) + h(t_2(z)) + h(t_3(z)) + h(t_{|C|-1}(z)) \\ &\geq h(t_1(z)) + 2h(t_3(z)) + h(L_{5,|C|-1}(z)) \\ &\geq h(U_{5,1}(z)) + 2h(L_{5,3}(z) + \varepsilon) + h(L_{5,|C|-1}(z)) \end{aligned}$$

(we use $t_1(z)U_{5,1}(z)$, $t_2(z) \leq t_3(z) \leq L_{5,3}(z)+\varepsilon$ and $t_{|C|-1}(z) \geq L_{5,|C|-1}(z)$), which implies the assertion since $h(t)$ is decreasing in $(-\infty, \alpha_1)$ and increasing in $(\alpha_2, +\infty)$. $\square$

After the optimization of a and b we can still have check$(z) \geq 0$ (converse to the inequality from Lemma 8′). Then we continue with a recursive procedure which replaces α_0 with $U_{5,1}(z)$ whenever possible in Lemma 7 and again turn to Lemma 8′ with better $U_{5,1}(z)$ and $L_{5,|C|-1}(z)$. With $\varepsilon = 0.008$ this approach rules out the two remaining cases $n = 7$, $|C| = 63$ and $n = 8$, $|C| = 81$.

Case 1.2. Let us have $t_3(z) \geq L_{5,3}(z)+\varepsilon$, where $\varepsilon = 0.008$ as above. We have the analog of Lemma 7 for obtaining to a better upper bound on $t_1(z)$.

Lemma 7′. *We have $t_1(z) \leq U_{5,1}(z)$, where $U_{5,1}(z)$ is the smallest root of the equation $f(t) = (\rho_0|C|-1)f(\alpha_0) - f(L_{5,|C|-1}(z))$ and $f(t) = (t - L_{5,3}(z) - \varepsilon)g(t)$.*

Proof. Use (2) with $f(t)$ for z and C. We have

$$\begin{aligned}
\rho_0|C|f(\alpha_0) &= f_0|C| - f(1) = \sum_{i=1}^{|C|-1} f(t_i(z)) \\
&\geq f(t_1(z)) + f(t_2(z)) + f(t_{|C|-1}(z)) \\
&\geq f(t_1(z)) + f(\alpha_0) + f(L_{5,|C|-1}(z)),
\end{aligned}$$

which implies the assertion since $f(t)$ is increasing in $(-\infty, \alpha_0)$. $\square$

We check for existence by Lemma 8 for the point x. After finding the optimal values of a and b we can still have the converse inequality, i.e. check$(x) \geq 0$. Then we continue with a recursive procedure which replaces α_0 with $U_{5,1}(z)$ whenever possible in Lemma 7′ and again turn to Lemma 8 with better $U_{5,1}(z)$ and $L_{5,|C|-1}(x)$. This rules out the two remaining cases $n = 7$, $|C| = 63$ and $n = 8$, $|C| = 81$.

Thus we finally have obtained the nonexistence of all 42 designs under consideration assuming $t_2(z) \in [\alpha_0, U_{5,2}(z)]$.

4.2 Case 2: $t_2(z) \in [t_1(z), \alpha_0]$

We have $t_2(z) \in [t_1(z), \alpha_0] \subseteq [L_{5,1}(z), U_{5,2}(z) := \alpha_0]$. We can not obtain good bounds $t_1(z) \leq U_{5,1}(z)$ at this point. This is why we start with investigation of the location of $t_3(z)$ with respect to α'.

Case 2.1. Let us have $t_3(z) \in [L_{5,3}(z), \alpha']$. We start with new lower bounds on $t_2(z)$ and $t_3(z)$.

Lemma 9. *We have $t_2(z) \geq L_{5,2}(z)$, where $L_{5,2}(z)$ is the smallest root of the equation $2g(t) = \rho_0|C|g(\alpha_0) - g(\alpha') - g(L_{5,|C|-1}(z))$.*

Proof. Use (2) with $g(t)$ for z and C. We have

$$\begin{aligned}\rho_0 g(\alpha_0)|C| &= g_0|C| - g(1) = \sum_{i=1}^{|C|-1} g(t_i(z)) \\ &\geq g(t_1(z)) + g(t_2(z)) + g(t_3(z)) + g(t_{|C|-1}(z)) \\ &\geq 2g(t_2(z)) + g(\alpha') + g(L_{5,|C|-1}(z)),\end{aligned}$$

which implies the assertion since $g(t)$ is decreasing in $(-\infty, \alpha_1)$. $\square$

Lemma 10. *We have* $t_3(z) \geq \min\{L_{5,3}(z), \alpha_1\}$, *where* $L_{5,3}(z)$ *is the smallest root of the equation* $g(t) = (\rho_0|C| - 2)g(\alpha_0) - g(L_{5,|C|-1}(z))$.

Proof. Using $t_2(z) \leq \alpha_0 = U_{5,2}(z)$ as in Lemma 6 we have

$$\begin{aligned}\rho_0 g(\alpha_0)|C| &= g_0|C| - g(1) = \sum_{i=1}^{|C|-1} g(t_i(z)) \\ &\geq g(t_1(z)) + g(t_2(z)) + g(t_3(z)) + g(t_{|C|-1}(z)) \\ &\geq 2g(\alpha_0) + g(t_3(z)) + g(L_{5,|C|-1}(z)),\end{aligned}$$

which implies the assertion since $g(t)$ is decreasing in $(-\infty, \alpha_1)$. $\square$

Remark. A new better bound $t_1(z) \geq L_{5,1}(z)$ can be obtained but we have not found its applications.

In all cases we have $L_{5,2}(z) \leq t_2(z) \leq \alpha_0 \leq L_{5,3}(z) \leq t_3(z) \leq \alpha'$ which seems to be a strong restriction.

Example 1. (Continued) We have $\alpha' = -0.680699$. Then Lemmas 9-10 give $t_2(z) \geq L_{5,2}(z) = -0.858038$ and $t_3(z) \geq L_{5,3}(z) = -0.769776$. $\diamond$

Example 2. (Continued) Analogously, we have $\alpha' = -0.664851$ and $t_2(z) \geq L_{5,2}(z) = -0.860278$ and $t_3(z) \geq L_{5,3}(z) = -0.798687$. $\diamond$

Now, we are in a position to obtain a upper bound $t_1(z) \leq U_{5,1}(z)$ as required by (6).

Lemma 11. *We have* $t_1(z) \leq U_{5,1}(z)$, *where* $U_{5,1}(z)$ *is the smallest root of the equation* $f(t) = -f(L_{5,2}(z)) - f(L_{5,3}(z)) - f(L_{5,|C|-1}(z))$, *where* $f(t) = (t - \alpha_0)g(t)$.

Proof. Use (2) with $f(t)$ for z and C. We have

$$\begin{aligned}0 &= f_0|C| - f(1) = \sum_{i=1}^{|C|-1} f(t_i(z)) \geq f(t_1(z)) + f(t_2(z)) + f(t_3(z)) + f(t_{|C|-1}(z)) \\ &\geq f(t_1(z)) + f(L_{5,2}(z)) + f(L_{5,3}(z)) + f(L_{5,|C|-1}(z)),\end{aligned}$$

which implies the assertion since $f(t)$ is increasing in $(-\infty, \alpha_0)$. We note the inequality $f(t_3(z)) \geq f(L_{5,3}(z))$ which follows by $t_3(z) \in [L_{5,3}(z), \alpha']$ and explains our choice to work with α'. $\square$

Lemma 12. *If there exist* $a, b \in [\alpha', \alpha_2]$ *such that* $\mathrm{check}(z) := h_0|C| - h(1) - h(U_{5,1}(z)) - h(\alpha_0) - h(\alpha') - h(L_{5,|C|-1}(z)) < 0$*, where* $h(t) = (t-a)^2(t-b)^2$*, then* C *does not exist.*

Proof. Use (2) with $h(t)$, z and C. We have

$$\begin{aligned} h_0|C| - h(1) &= \sum_{i=1}^{|C|-1} h(t_i(z)) \geq h(t_1(z)) + h(t_2(z)) + h(t_3(z)) + h(t_{|C|-1}(z)) \\ &\geq h(U_{5,1}(z)) + h(\alpha_0) + h(\alpha') + h(L_{5,|C|-1}(z)), \end{aligned}$$

which implies the assertion. $\square$

As in Case 1 we apply a recursive procedure. We come back consecutively to Lemmas 9-11 and check(z) for existence by Lemma 12, while check$(z) \geq 0$.

Example 1. (Continued) We have $\alpha' = -0.680699$. The first step gives $t_2(z) \geq L_{5,2}(z) = -0.858038$, $t_3(z) \geq L_{5,3}(z) = -0.769776$, $t_1(z) \leq U_{5,1}(z) = -0.848575$ and check$(z) = 0.079302 > 0$ but the second step gives $t_2(z) \geq L_{5,2}(z) = -0.856549$, $t_3(z) \geq L_{5,3}(z) = -0.739912$, $t_1(z) \leq U_{5,1}(z) = -0.874519$ and check$(z) = -0.009398 < 0$. Therefore 5-designs on $\mathbb{S}^{10}$ with 147 points such that $t_2(z) \in [t_1(z), \alpha_0]$ and $t_3(z) \in [L_{5,3}(z), \alpha']$ do not exist. $\diamond$

Example 2. (Continued) We have $\alpha' = -0.664851$. The first step gives $t_2(z) \geq L_{5,2}(z) = -0.860278$, $t_3(z) \geq L_{5,3}(z) = -0.798687$ and $t_1(z) \leq U_{5,1}(z) = -0.775811$. This is one of the bad cases when $U_{5,1}(z) > \alpha_0$. $\diamond$

The last procedure rules out 18 out of all 42 cases. The remaining 24 cases (including Example 2) are resolved by a precise consideration how close is $t_3(z)$ to $L_{5,3}(z)$ as in the end of Case 1. More precisely we have the following two subcases.

Case 2.1.1. Let us have $t_3(z) \in [L_{5,3}(z), L_{5,3}(z)+\varepsilon]$, where $\varepsilon > 0$ is such that $L_{5,3}(z)+\varepsilon < \alpha'$. Now we have new upper bound $t_3(z) \leq U_{5,3}(z) = L_{5,3}(z) + \varepsilon$ and analogs of Lemma 11 and Lemma 12.

Lemma 11′. *We have* $t_1(z) \leq U_{5,1}(z)$*, where* $U_{5,1}(z)$ *is the smallest root of the equation* $f(t) = -f(L_{5,2}(z)) - f(L_{5,3}(z)) - f(L_{5,|C|-1}(z))$*, where* $f(t) = (t-\alpha_0)g(t)$*.*

Proof. Use (2) with $f(t)$ for z and C. We have

$$\begin{aligned} 0 &= f_0|C| - f(1) = \sum_{i=1}^{|C|-1} f(t_i(z)) \geq f(t_1(z)) + f(t_2(z)) + f(t_3(z)) + f(t_{|C|-1}(z)) \\ &\geq f(t_1(z)) + f(L_{5,2}(z)) + f(L_{5,3}(z)) + f(L_{5,|C|-1}(z)), \end{aligned}$$

which implies the assertion since $f(t)$ is increasing in $(-\infty, \alpha_0)$. $\square$

Lemma 12′. *If there exist* $a, b \in [\alpha', \alpha_2]$ *such that* $\mathrm{check}(z) := h_0|C| - h(1) - h(U_{5,1}(z)) - h(\alpha_0) - h(U_{5,3}(z)) - h(L_{5,|C|-1}(z)) < 0$, *where* $h(t) = (t-a)^2(t-b)^2$, *then* C *does not exist.*

Proof. Use (2) with $h(t)$, z and C. We have

$$\begin{aligned} h_0|C| - h(1) &= \sum_{i=1}^{|C|-1} h(t_i(z)) \geq h(t_1(z)) + h(t_2(z)) + h(t_3(z)) + h(t_{|C|-1}(z)) \\ &\geq h(U_{5,1}(z)) + h(\alpha_0) + h(U_{5,3}(z)) + h(L_{5,|C|-1}(z)), \end{aligned}$$

which implies the assertion. $\square$

A recursive procedure (as described above) using Lemmas 9-10, $11', 12'$ rules out the remaining 24 cases when $t_2(z) \in [t_1(z), \alpha_0]$ and $t_3(z) \in [L_{5,3}(z), L_{5,3}(z) + \varepsilon]$. For 22 of them $\varepsilon = 0.01$ works. For the remaining two cases ($n = 19$, $|C| = 427$) and ($n = 21$, $|C| = 519$), we need $\varepsilon = 0.008$ and $\varepsilon = 0.007$ respectively.

Example 2. (Continued) We have $\alpha' = -0.664851$ and $\varepsilon = 0.01$. The first step gives $t_2(z) \geq L_{5,2}(z) = -0.860278$, $t_3(z) \geq L_{5,3}(z) = -0.798687$, $t_3(z) \leq U_{5,3}(z) = L_{5,3}(z) + \varepsilon = -0.788687$, $t_1(z) \leq U_{5,1}(z) = -0.826848$ and $\mathrm{check}(z) = -0.006815 < 0$. Therefore 5-designs on $\mathbb{S}^{16}$ with 343 points such that $t_2(z) \in [t_1(z), \alpha_0]$ and $t_3(z) \in [L_{5,3}(z), L_{5,3}(z) + 0.01]$ do not exist. $\diamond$

Case 2.1.2. Let us have $t_3(z) \in [L_{5,3}(z) + \varepsilon, \alpha']$, where ε is as above. Then we apply the analog of Lemma 11 and check(z) by Lemma 12.

Lemma 11″. *We have* $t_1(z) \leq U_{5,1}(z)$, *where* $U_{5,1}(z)$ *is the smallest root of the equation* $f(t) = -f(L_{5,2}(z)) - f(L_{5,3}(z) + \varepsilon) - f(L_{5,|C|-1}(z))$, *where* $f(t) = (t - \alpha_0)g(t)$.

Proof. Use (2) with $f(t)$ for z and C. We have

$$\begin{aligned} 0 &= f_0|C| - f(1) = \sum_{i=1}^{|C|-1} f(t_i(z)) \geq f(t_1(z)) + f(t_2(z)) + f(t_3(z)) + f(t_{|C|-1}(z)) \\ &\geq f(t_1(z)) + f(L_{5,2}(z)) + f(L_{5,3}(z) + \varepsilon) + f(L_{5,|C|-1}(z)), \end{aligned}$$

which implies the assertion since $f(t)$ is increasing in $(-\infty, \alpha_0)$. $\square$

A recursive procedure with Lemmas 9-10, $11''$ and 12 resolves all remaining 24 cases when $t_2(z) \in [t_1(z), \alpha_0]$ and $t_3(z) \in [L_{5,3}(z) + \varepsilon, \alpha']$.

Example 2. (Continued) Again, we have $\alpha' = -0.664851$ and $\varepsilon = 0.01$. After six steps we obtain $t_2(z) \geq L_{5,2}(z) = -0.852473$, $t_3(z) \geq L_{5,3}(z) = -0.694411$, $t_1(z) \leq U_{5,1}(z) =$

-0.885379 and $\mathrm{check}(z) = -0.028429 < 0$. Therefore 5-designs on $\mathbb{S}^{16}$ with 343 points such that $t_2(z) \in [t_1(z), \alpha_0]$ and $t_3(z) \in [L_{5,3}(z) + 0.01, \alpha']$ do not exist. $\diamond$

This ends Case 2.1 with a nonexistence proof for all 42 designs under consideration under the assumption $t_2(z) \in [t_1(z), \alpha_0]$ and $t_3(z) \in [L_{5,3}(z), \alpha']$.

Case 2.2. Let us have $t_3(z) > \alpha'$, i.e. $t_3(z) \geq L_{5,3}(z) = \alpha'$. We obtain immediately a upper bound $t_1(z) \leq U_{5,1}(z)$ as required by (6).

Lemma 13. *We have $t_1(z) \leq U_{5,1}(z)$, where $U_{5,1}(z)$ is the smallest root of the equation $2f(t) = \rho_0|C|f(\alpha_0) - f(L_{5,|C|-1}(z))$, where $f(t) = (t - \alpha')g(t)$.*

Proof. Use (2) with $f(t)$ for z and C. We have

$$\begin{aligned} \rho_0|C|f(\alpha_0) &= f_0|C| - f(1) = \sum_{i=1}^{|C|-1} f(t_i(z)) \\ &\geq f(t_1(z)) + f(t_2(z)) + f(t_{|C|-1}(z)) \geq 2f(t_1(z)) + f(L_{5,|C|-1}(z)), \end{aligned}$$

which implies the assertion. $\square$

We now apply check(x) by using the better bound

$$L_{5,|C|-1}(x) = U_{5,1}(z)\alpha_0 - \sqrt{(1 - U_{5,1}^2(z))(1 - \alpha_0^2)}$$

in Lemma 8 using $t_2(z) \leq \alpha_0 = U_{5,2}(z)$.

If $\mathrm{check}(x) \geq 0$ we continue with a recursive procedure which replaces α_0 with $U_{5,1}(z)$ whenever possible and again turn to check(x) with better $U_{5,1}(z)$ and $L_{5,|C|-1}(x)$.

Example 1. (Continued) We need eight steps to obtain $t_3(z) \geq \alpha' = -0.680699$, $t_1(z) \leq U_{5,1}(z) = -0.890144$, $L_{5,|C|-1}(x) = 0.401037$ and $\mathrm{check}(x) = -0.011722 < 0$. This completes the proof in the last case. Therefore there exist no 5-designs on $\mathbb{S}^{10}$ with 147 points. $\diamond$

Example 2. (Continued) Similarly, we have $\alpha' = -0.664851$ and $\varepsilon = 0.01$. We need twelve steps to obtain $t_1(z) \leq U_{5,1}(z) = -0.890753$, $L_{5,|C|-1}(x) = 0.359055$ and $\mathrm{check}(x) = -0.015316 < 0$. This completes the proof in the last case. Therefore there exist no 5-designs on $\mathbb{S}^{16}$ with 343 points. $\diamond$

This procedure rules out 36 out of all 42 cases. The last 6 cases are now ruled out by a precise consideration how close is $t_3(z)$ to α' and how close is $t_2(z)$ to α_0). We omit the details.

Acknowledgments. This research was partially supported by the Bulgarian NSF under Contract MM-1405/04.

References

[1] M. Abramowitz, I. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1965.

[2] B. Bajnok, Constructions of spherical 3-designs, *Graphs and Combinatorics* 14, 1998, 97-107.

[3] B. Bajnok, Constructions of spherical 3-designs, *Designs, Codes and Cryptography* 2000.

[4] S. P. Boumova, P. G. Boyvalenkov, D. P. Danev, New nonexistence results for spherical designs, in Constructive Theory of Sunctions (B. Bojanov, Ed.) Darba, Sofia 2003, 225-232.

[5] P.G. Boyvalenkov, S.P. Bumova, D.P. Danev, Necessary conditions for existence of some designs in polynomial metric spaces, *Europ. J. Combin.* 20, 1999, 213-225.

[6] P.G. Boyvalenkov, D.P. Danev, S.I. Nikova, Nonexistence of certain spherical designs of odd strengths and cardinalities, *Discr. Comp. Geom.* 21, 1999, 143-156.

[7] J. H. Conway, N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, Springer-Verlag, New York 1988.

[8] P. Delsarte, J.-M. Goethals, J. J. Seidel, Spherical codes and designs, *Geom. Dedicata* 6, 1977, 363-388.

[9] R. H. Hardin, N. J. A. Sloane, McLaren's improved snub cube and other new spherical designs in three dimensions, *Discr. Comp. Geom.* 15, 1996, 429-441.

[10] V. I. Levenshtein, Universal bounds for codes and designs, Chapter 6 (499-648) in *Handbook of Coding Theory*, Eds. V.Pless and W.C.Huffman, Elsevier Science B.V., 1998.

[11] B. Reznick, Some constructions of spherical 5-designs, *Lin. Alg. Appl.* 226/228, 1995, 163-196.