Minimization of a strictly convex separable function subject to a convex inequality constraint or linear equality constraints and bounds on the variables

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Abstract

In this paper, we consider the problem of minimizing a strictly convex separable function over a feasible region defined by a convex inequality constraint and two-sided bounds on the variables (box constraints). Also, the convex separable program with a strictly convex objective function subject to linear equality constraints and bounded variables is considered. These problems are interesting from both theoretical and practical point of view because they arise in some mathematical programming problems and in various practical problems. Characterization theorems (necessary and sufficient conditions) for the optimal solution to the considered problems are proved. Some illustrative examples are also presented.

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1 Introduction

Consider the following strictly convex separable program with strictly convex inequality constraint and bounded variables
(SCS)

$$\min \left\{ c(\mathbf{x}) = \sum_{j \in J} c_j(x_j) \right\} \tag{1.1}$$

subject to

$$d(\mathbf{x}) \equiv \sum_{j \in J} d_j(x_j) \equiv \sum_{j \in J} d_j x_j^p \leq \alpha \tag{1.2}$$

$$a_j \leq x_j \leq b_j, \ j \in J, \tag{1.3}$$

where $c_j(x_j)$ are strictly convex differentiable functions, $c_j'(x_j) > 0$, $d_j > 0$, $x_j > 0$ for every $j \in J$, $p > 1$, $\mathbf{x} = (x_j)_{j \in J}$, and $J \stackrel{\text{def}}{=} \{1, \ldots, n\}$.

Functions $d_j(x_j), j \in J$, are also stricly convex because $d_j''(x_j) \equiv p(p-1)x_j^{p-2} > 0, j \in J$, under the assumptions. In particular, since $x_j > 0, j \in J$, then $a_j > 0, b_j > 0, j \in J$.

Feasible region X, defined by (1.2) – (1.3), is intersection of the half-space (1.2) and the $n-$dimensional box (1.3). Therefore X is a convex set.

One of the most important problems of the form (SCS) is the program with $c_j(x_j) = c_j x_j^q, j \in J$. Functions $c_j(x_j) = c_j x_j^q$ are strictly convex for $c_j > 0, x_j > 0, j \in J, q > 1$, and the assumption $c_j'(x_j) \equiv q c_j x_j^{q-1} > 0, j \in J$, is also satisfied for $c_j > 0, x_j > 0, j \in J, q > 1$ in this case.

Problem (SCS) is a convex separable programming problem because the objective function and constraint function are convex (moreover, strictly convex) and separable (that is, these functions can be expressed as the sums of single-variable functions). Because of the strict convexity, if problem (SCS) is solvable, its solution is unique.

Also, consider the following convex separable program with strictly convex objective function subject to linear equality constraints and bounded variables

($C_m^=$)

$$\min \left\{ c(\mathbf{x}) = \sum_{j \in J} c_j(x_j) \right\} \tag{1.4}$$

subject to

$$D\mathbf{x} = \boldsymbol{\alpha} \tag{1.5}$$

$$\mathbf{a} \le \mathbf{x} \le \mathbf{b}, \tag{1.6}$$

where $c_j(x_j)$ are strictly convex differentiable functions, $j \in J$, $D = (d_{ij}) \in \mathbb{R}^{m \times n}$, $\boldsymbol{\alpha} \in \mathbb{R}^m$, $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$.

The feasible region $X^=$, defined by (1.5) – (1.6), is an intersection of m hyperplanes (1.5) and the box (1.6). Therefore $X^=$ is a convex set.

Problem ($C_m^=$) is also a convex separable programming problem because the objective function and constraint functions are convex and separable. Because of the strict convexity of objective function, if problem ($C_m^=$) is solvable, its solution is unique.

Problems (SCS) and ($C_m^=$), defined by (1.1) – (1.3) and (1.4) – (1.6), respectively, and related to them, arise in production planning and scheduling [18], in allocation of resources [1, 5, 6, 18, 20], in decision making [1, 5, 6, 20], in the theory of search, in facility location [7, 9, 20], etc.

Problems like (SCS) and ($C_m^=$) are subject of intensive study. Related problems and methods for them are considered in [1 – 20]. Algorithms for resource allocation problems are proposed in [1, 5, 6, 18, 20], and algorithms for facility location problems are suggested in [7], etc. Singly constrained quadratic programs with bounded variables are considered in [3, 4], and some separable programs are considered and methods for solving them are suggested in [2, 8 – 19], etc.

This paper is devoted to solution of problems (SCS) and ($C_m^=$). The paper is organized as follows. In Section 2, characterization theorems (necessary and sufficient conditions) for the optimal solution to problems (SCS) and ($C_m^=$) are proved. Due to the specific form of the optimal solution to problem (SCS), this solution can be obtained directly and it is not necessary to develop iterative algorithms for solving problem (SCS). In Section 3 we present results of two simple illustrative examples for both problem (SCS) and problem ($C_m^=$).

2 Characterization theorems

2.1 Problem (SCS)

Consider problem (SCS) defined by (1.1) – (1.3).

Suppose that following assumptions are satisfied.

(A1) $a_j \le b_j$ for all $j \in J$. If $a_k = b_k$ for some $k \in J$, then the value $x_k := a_k = b_k$ is determined in advance.

(A2) $\sum_{j\in J} d_j a_j^p \le \alpha$. Otherwise the constraints (1.2) – (1.3) are inconsistent and $X = \emptyset$, where the feasible region X is defined by (1.2) – (1.3).

The Lagrangian for problem (SCS) is

$$L(\mathbf{x}, \mathbf{u}, \mathbf{v}, \lambda) = \sum_{j\in J} c_j(x_j) + \lambda\Big(\sum_{j\in J} d_j x_j^p - \alpha\Big) + \sum_{j\in J} u_j(a_j - x_j) + \sum_{j\in J} v_j(x_j - b_j), \quad (2.1)$$

where $\lambda \in \mathbb{R}^1_+; \mathbf{u}, \mathbf{v} \in \mathbb{R}^n_+$, and $\mathbb{R}^n_+$ consists of all vectors with n real nonnegative components.

The Karush-Kuhn-Tucker (KKT) necessary and sufficient optimality conditions for the minimum solution $\mathbf{x}^* = (x_j^*)_{j\in J}$ to problem (SCS) are

$$c_j(x_j^*) + \lambda p d_j.(x_j^*)^{p-1} - u_j + v_j = 0, \quad j \in J \quad (2.2)$$

$$u_j(a_j - x_j^*) = 0, \quad j \in J \quad (2.3)$$

$$v_j(x_j^* - b_j) = 0, \quad j \in J \quad (2.4)$$

$$\lambda\Big(\sum_{j\in J} d_j.(x_j^*)^p - \alpha\Big) = 0, \quad \lambda \in \mathbb{R}^1_+ \quad (2.5)$$

$$\sum_{j\in J} d_j.(x_j^*)^p \le \alpha \quad (2.6)$$

$$a_j \le x_j^* \le b_j, \quad j \in J \quad (2.7)$$

$$u_j \in \mathbb{R}^1_+,\ v_j \in \mathbb{R}^1_+, \quad j \in J, \quad (2.8)$$

where $\lambda, u_j, v_j,\ j \in J$, are the Lagrange multipliers associated with the constraints (1.2), $a_j \le x_j, x_j \le b_j,\ j \in J$, respectively. If $a_j = -\infty$ or $b_j = +\infty$ for some j, we do not consider the corresponding condition (2.3) ((2.4), respectively) and Lagrange multiplier u_j (v_j, respectively).

According to conditions (2.2) – (2.8), $\lambda \ge 0$, $u_j \ge 0$, $v_j \ge 0, j \in J$, and complementary conditions (2.3), (2.4), (2.5) must be satisfied. In order to find $x_j^*, j \in J$, from system (2.2) – (2.8), we have to consider all possible cases for λ, u_j, v_j: all λ, u_j, v_j equal to 0; all λ, u_j, v_j different from 0; some of them equal to 0 and some of them different from 0. The number of these cases is 2^{2n+1}, where $2n+1$ is the number of all $\lambda, u_j, v_j, j \in J$, $|J| = n$. This is an enormous number of cases, especially for large-scale problems. Moreover, in each case we have to solve a large-scale system of nonlinear equations in $x_j^*, \lambda, u_j, v_j, j \in J$. Therefore the *direct* application of the KKT theorem, using explicit enumeration of all possible cases, for solving large-scale problems of the considered form would not be effective. That is why, we need efficient methods for solving the considered problem.

Theorem 2.1 gives a characterization of the optimal solution to problem (SCS). Its proof is based on the KKT theorem.

Theorem 2. 1 (Characterization of the optimal solution to problem (SCS)) *The point* $\mathbf{x}^* = (x_j^*)_{j \in J}$ *is the optimal solution to problem* (SCS) *if and only if*

$$\mathbf{x}^* = (a_1, \ldots, a_n). \tag{2.9}$$

Proof. *Necessity.* Let $\mathbf{x}^* = (x_j^*)_{j \in J}$ be the optimal solution to (SCS). Then there exist nonnegative constants $\lambda, u_j, v_j,\ j \in J$, such that KKT conditions (2.2) – (2.8) are satisfied. Consider both possible cases for λ.

(1) Let $\lambda > 0$. Then system (2.2) – (2.8) becomes (2.2), (2.3), (2.4), (2.7), (2.8) and

$$\sum_{j \in J} d_j.(x_j^*)^p = \alpha, \tag{2.10}$$

that is, the inequality constraint (1.2) is satisfied with an equality for $x_j^*, j \in J$, in this case.

(a) If $x_j^* = a_j$, then $u_j \geq 0$, and $v_j = 0$ according to (2.4). Therefore (2.2) implies $c_j'(x_j^*) = u_j - \lambda p d_j.(x_j^*)^{p-1} \geq -\lambda p d_j.(x_j^*)^{p-1}$. Since $d_j > 0,\ x_j^* > 0, j \in J,\ p > 1$, then

$$\lambda \geq -\frac{c_j'(x_j^*)}{p d_j.(x_j^*)^{p-1}} \equiv -\frac{c_j'(a_j)}{p d_j a_j^{p-1}}. \tag{2.11}$$

Because $c_j'(.) > 0, d_j > 0, a_j > 0, j \in J, p > 1\ (> 0)$, and $\lambda \geq 0$, the inequality (2.11) is always satisfied.

(b) If $x_j^* = b_j$, then $u_j = 0$ according to (2.3), and $v_j \geq 0$. Therefore (2.2) implies $c_j'(x_j^*) = -v_j - \lambda p d_j.(x_j^*)^{p-1} \leq -\lambda p d_j.(x_j^*)^{p-1}$. Hence

$$\lambda \leq -\frac{c_j'(x_j^*)}{p d_j.(x_j^*)^{p-1}} \equiv -\frac{c_j'(b_j)}{p d_j b_j^{p-1}}. \tag{2.12}$$

Since $c_j'(.) > 0, d_j > 0, b_j > 0, j \in J, p > 1\ (> 0)$, then

$$-\frac{c_j'(b_j)}{p d_j b_j^{p-1}} < 0, \tag{2.13}$$

and since λ must be nonnegative, from (2.12) and (2.13) it is obvious that this case *is impossible*.

(c) If $a_j < x_j^* < b_j$, then $u_j = v_j = 0$ according to (2.3) and (2.4). Therefore (2.2) implies

$$-c_j'(x_j^*) = \lambda p d_j.(x_j^*)^{p-1}. \tag{2.14}$$

Since $c_j'(.) > 0, d_j > 0,\ x_j^* > 0, j \in J,\ p > 1$ by the assumption, and $\lambda \geq 0$, from (2.14) it follows that this case *is impossible*.

(2) Let $\lambda = 0$. Then system (2.2) – (2.8) becomes

$$c_j'(x_j^*) - u_j + v_j = 0,\ j \in J \tag{2.15}$$

and (2.3), (2.4), (2.6), (2.7), (2.8).

(a) If $x_j^* = a_j$, then $u_j \geq 0, v_j = 0$. Therefore

$$c_j'(a_j) \equiv c_j'(x_j^*) = u_j \geq 0. \tag{2.16}$$

Since $c_j'(.) > 0$ by the assumption, then (2.16) is always satisfied.

(b) If $x_j^* = b_j$, then $u_j = 0, v_j \geq 0$. Therefore

$$c_j'(b_j) \equiv c_j'(x_j^*) = -v_j \leq 0. \tag{2.17}$$

Since $c_j'(.) > 0$, this case *is impossible.*

(c) If $a_j < x_j^* < b_j$, then $u_j = v_j = 0$. Therefore $c_j'(x_j^*) = 0$, and since $c_j'(.) > 0$, this case *is impossible.*

As we have proved, in both cases (1) and (2), only subcase (a) is possible, that is, $\mathbf{x}^* = (a_1, \ldots, a_n)$. The "necessity" part is proved.

Sufficiency. Conversely, let $\mathbf{x}^* = (a_1, \ldots, a_n)$. Obviously $\mathbf{x}^* \in X$, where X is defined by (1.2) – (1.3).

Set

$$u_j = c_j'(a_j) + \lambda p d_j a_j^{p-1} \quad (\geq 0 \text{ under the assumptions}), \quad v_j = 0. \tag{2.18}$$

By using these expressions, it is easy to check that $x_j^*, \lambda, u_j, v_j, j \in J$, satisfy conditions (2.2), (2.3), (2.4), (2.5), (2.8); conditions (2.6) and (2.7) are also satisfied because $\mathbf{x}^* = (a_1, \ldots, a_n) \in X$.

Since (2.2) – (2.8) are necessary and sufficient conditions for an optimal solution to the convex minimization problem (SCS), then $\mathbf{x}^*$ is an optimal solution to this problem, and since $c(\mathbf{x})$ is a strictly convex function as the sum of strictly convex functions, this optimal solution is unique. ■

2.2 Problem $(C_m^=)$

Denote by $P_c(D, \boldsymbol{\alpha}, \mathbf{a}, \mathbf{b})$ the solution to problem $(C_m^=)$. Since $c(\mathbf{x})$ is strictly convex as the sum of strictly convex functions, then $P_c(D, \boldsymbol{\alpha}, \mathbf{a}, \mathbf{b})$ is uniquely defined, that is, there is at most one minimum which is both local and global.

Denote $\mathbf{y} = [\mathbf{x}]_\mathbf{a}^\mathbf{b}$ where $y_j = \min\{\max\{x_j, a_j\}, b_j\}$ for each $j \in J$.

The KKT conditions for $\mathbf{x}^* \in \mathbb{R}^n$ to be minimum solution to problem $(C_m^=)$ are

$$D\mathbf{x}^* = \boldsymbol{\alpha} \tag{2.19}$$

$$\mathbf{a} \leq \mathbf{x}^* \tag{2.20}$$

$$\mathbf{x}^* \leq \mathbf{b} \tag{2.21}$$

$$\mathbf{c}'(\mathbf{x}^*) + D^T\boldsymbol{\lambda} - \mathbf{u} + \mathbf{v} = \mathbf{0} \tag{2.22}$$

$$u_j(a_j - x_j^*) = 0, \quad j \in J \tag{2.23}$$

$$v_j(x_j^* - b_j) = 0, \quad j \in J \tag{2.24}$$

$$\mathbf{u} \geq \mathbf{0} \tag{2.25}$$

$$\mathbf{v} \geq \mathbf{0}, \tag{2.26}$$

where $\boldsymbol{\lambda} \in \mathbb{R}^m$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_+$ are the Lagrange multipliers associated with (1.5) and the two inequalities of (1.6), respectively.

The map $\mathbf{c}' \equiv \nabla c : \mathbb{R}^n \to \mathbb{R}^n$ is strict monotone increasing since c is a strictly convex function. Therefore $(\nabla c)^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is well-defined.

Theorem 2. 2 (Characterization of the optimal solution to problem $(C_m^=)$) *Let $c : \mathbb{R}^n \to \mathbb{R}$ be separable, differentiable and strictly convex. Then*

$$\{P_c(D, \boldsymbol{\alpha}, \mathbf{a}, \mathbf{b})\} = \left\{ (\mathbf{c}')^{-1} \left[-D^T \mathbf{t} \right]_{\mathbf{c}'(\mathbf{a})}^{\mathbf{c}'(\mathbf{b})} : \mathbf{t} \in \mathbb{R}^m \right\}, \tag{2.27}$$

where $D, \boldsymbol{\alpha}, \mathbf{a}, \mathbf{b}$ are defined above.

Proof. Relation (2.27) is proved by two-way inclusion.

(1) Let $\mathbf{x}^* = P_c(D, \boldsymbol{\alpha}, \mathbf{a}, \mathbf{b})$ for some $\boldsymbol{\alpha} \in \mathbb{R}^m$. Then there exist $\boldsymbol{\lambda} \in \mathbb{R}^m$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_+$ satisfying the KKT conditions (2.19) – (2.26) together with this $\mathbf{x}^*$.

From (2.22) it follows that

$$D^T \boldsymbol{\lambda} = -\mathbf{c}'(\mathbf{x}^*) + \mathbf{u} - \mathbf{v}, \tag{2.28}$$

that is,

$$\langle \mathbf{D}_j, \boldsymbol{\lambda} \rangle = -c_j'(x_j^*) + u_j - v_j \tag{2.29}$$

for each $j \in J$ where $\langle \mathbf{x}, \mathbf{y} \rangle$ denotes the inner (scalar) product of $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.

If $\langle \mathbf{D}_j, \boldsymbol{\lambda} \rangle > -c_j'(x_j^*)$, then $u_j > v_j \geq 0$, so $x_j^* = a_j$ according to (2.23), that is,

$$\langle \mathbf{D}_j, \boldsymbol{\lambda} \rangle > -c_j'(x_j^*) \quad \text{implies} \quad x_j^* = a_j. \tag{2.30}$$

Similarly, if $\langle \mathbf{D}_j, \boldsymbol{\lambda} \rangle < -c_j'(x_j^*)$, then $v_j > u_j \geq 0$, so $x_j^* = b_j$ according to (2.24), that is,

$$\langle \mathbf{D}_j, \boldsymbol{\lambda} \rangle < -c_j'(x_j^*) \quad \text{implies} \quad x_j^* = b_j. \tag{2.31}$$

Since $a_j \leq b_j, j \in J$, by assumption, we have three cases to consider.

Case 1. $\langle \mathbf{D}_j, \boldsymbol{\lambda} \rangle > -c_j'(a_j)$.

Then $\langle \mathbf{D}_j, \boldsymbol{\lambda} \rangle > -c_j'(x_j^*)$ according to (2.20) and the monotonicity of c_j'. Hence $x_j^* = a_j$ in accordance with (2.30).

Case 2. $\langle \mathbf{D}_j, \boldsymbol{\lambda} \rangle < -c_j'(b_j)$.

Then $\langle \mathbf{D}_j, \boldsymbol{\lambda} \rangle < -c_j'(x_j^*)$ according to (2.21) and the monotonicity of c_j'. Hence $x_j^* = b_j$ in accordance with (2.31).

Case 3. $-c_j'(b_j) \leq \langle \mathbf{D}_j, \boldsymbol{\lambda} \rangle \leq -c_j'(a_j)$.

If $\langle \mathbf{D}_j, \boldsymbol{\lambda} \rangle < -c_j'(x_j^*)$, then $x_j^* = b_j$ according to (2.31). Therefore $\langle \mathbf{D}_j, \boldsymbol{\lambda} \rangle \geq -c_j'(x_j^*)$ because $\langle \mathbf{D}_j, \boldsymbol{\lambda} \rangle \geq -c_j'(b_j)$ by the assumption of Case 3, a contradiction. Similarly, if we assume that $\langle \mathbf{D}_j, \boldsymbol{\lambda} \rangle > -c_j'(x_j^*)$ strictly, this would imply $x_j^* = a_j$ according to (2.30) and $\langle \mathbf{D}_j, \boldsymbol{\lambda} \rangle \leq -c_j'(x_j^*)$, a contradiction.

Then $\langle \mathbf{D}_j, \boldsymbol{\lambda} \rangle = -c_j'(x_j^*)$, so it follows that $x_j^* = (c_j')^{-1}(-\langle \mathbf{D}_j, \boldsymbol{\lambda} \rangle)$.

In the three cases considered, we have

$$x_j^* = (c_j')^{-1}\Big[-\langle \mathbf{D}_j, \boldsymbol{\lambda}\rangle\Big]_{c_j'(a_j)}^{c_j'(b_j)}. \tag{2.32}$$

Hence $\mathbf{x}^* = (\mathbf{c}')^{-1}\Big[-D^T\boldsymbol{\lambda}\Big]_{\mathbf{c}'(\mathbf{a})}^{\mathbf{c}'(\mathbf{b})}$, that is,

$$\{P_c(D, \boldsymbol{\alpha}, \mathbf{a}, \mathbf{b})\} \subseteq \left\{(\mathbf{c}')^{-1}\Big[-D^T\mathbf{t}\Big]_{\mathbf{c}'(\mathbf{a})}^{\mathbf{c}'(\mathbf{b})} : \mathbf{t} \in \mathbb{R}^m\right\}. \tag{2.33}$$

(2) Conversely, suppose that $\mathbf{x}^* \in \mathbb{R}^n$ and $\mathbf{x}^* = (\mathbf{c}')^{-1}\Big[-D^T\mathbf{t}\Big]_{\mathbf{c}'(\mathbf{a})}^{\mathbf{c}'(\mathbf{b})}$ for some $\mathbf{t} \in \mathbb{R}^m$. Set:

$\boldsymbol{\alpha} = D(\mathbf{c}')^{-1}\Big[-D^T\mathbf{t}\Big]_{\mathbf{c}'(\mathbf{a})}^{\mathbf{c}'(\mathbf{b})}$
$\boldsymbol{\lambda} = \mathbf{t}$
$\mathbf{u} = \mathbf{c}'(\mathbf{a}) + D^T\mathbf{t}$
$\mathbf{v} = -\mathbf{c}'(\mathbf{b}) - D^T\mathbf{t}$.

We have to prove that $\mathbf{x}^*, \boldsymbol{\alpha}, \boldsymbol{\lambda}, \mathbf{u}, \mathbf{v}$ satisfy the KKT conditions (2.19) – (2.26).

Obviously $\mathbf{x}^*$ and $\boldsymbol{\alpha}$ satisfy (2.19), $\mathbf{x}^*$ satisfies (2.20) and (2.21) (these are (1.6) with $\mathbf{x} = \mathbf{x}^*$) according to definition of $[\mathbf{x}]_{\mathbf{a}}^{\mathbf{b}}$ and monotonicity of $\mathbf{c}'$.

In order to verify (2.22) – (2.26), we consider each $j \in J$. There are three possible cases.

Case 1. $\langle \mathbf{D}_j, \mathbf{t}\rangle > -c_j'(a_j)$.

Then $c_j'(a_j) + \langle \mathbf{D}_j, \mathbf{t}\rangle > 0$, and since $a_j \le b_j$, then $-c_j'(b_j) - \langle \mathbf{D}_j, \mathbf{t}\rangle < 0$. Therefore $x_j^* = a_j$, $\boldsymbol{\lambda} = \mathbf{t}$, $u_j = c_j'(a_j) + \langle \mathbf{D}_j, \mathbf{t}\rangle$, $v_j = 0$.

Case 2. $\langle \mathbf{D}_j, \mathbf{t}\rangle < -c_j'(b_j)$.

Then $-c_j'(b_j) - \langle \mathbf{D}_j, \mathbf{t}\rangle > 0$, and since $a_j \le b_j$, then $c_j'(a_j) + \langle \mathbf{D}_j, \mathbf{t}\rangle < 0$. Therefore $x_j^* = b_j$, $\boldsymbol{\lambda} = \mathbf{t}$, $u_j = 0$, $v_j = -c_j'(b_j) - \langle \mathbf{D}_j, \mathbf{t}\rangle$.

Case 3. $-c_j'(b_j) \le \langle \mathbf{D}_j, \mathbf{t}\rangle \le -c_j'(a_j)$.

Then $-c_j'(b_j) - \langle \mathbf{D}_j, \mathbf{t}\rangle \le 0$, $\langle \mathbf{D}_j, \mathbf{t}\rangle + c_j'(a_j) \le 0$. Therefore $x_j^* = (\mathbf{c}')_j^{-1}(-\langle \mathbf{D}_j, \mathbf{t}\rangle)$, $\boldsymbol{\lambda} = \mathbf{t}$, $u_j = v_j = 0$.

Obviously in each of the three cases, $x_j^*, u_j, v_j (j \in J), \boldsymbol{\lambda}$ satisfy (2.22) – (2.26) as well.

Therefore $\mathbf{x}^*, \boldsymbol{\alpha}, \boldsymbol{\lambda}, \mathbf{u}, \mathbf{v}$ satisfy the KKT conditions (2.19) – (2.26), so $\mathbf{x}^* \in P_c(D, \boldsymbol{\alpha}, \mathbf{a}, \mathbf{b})$ according to definition of $P_c(D, \boldsymbol{\alpha}, \mathbf{a}, \mathbf{b})$.

The two-way inclusion implies (2.27). ∎

Define the functions $\mathbf{x} : \mathbb{R}^m \to \mathbb{R}^n$, $\boldsymbol{\alpha} : \mathbb{R}^m \to \mathbb{R}^m$ by

$$\mathbf{x}(\mathbf{t}) = (\mathbf{c}')^{-1}\Big[-D^T\mathbf{t}\Big]_{\mathbf{c}'(\mathbf{a})}^{\mathbf{c}'(\mathbf{b})} \tag{2.34}$$

$$\boldsymbol{\alpha}(\mathbf{t}) = D(\mathbf{c}')^{-1}\Big[-D^T\mathbf{t}\Big]_{\mathbf{c}'(\mathbf{a})}^{\mathbf{c}'(\mathbf{b})} \tag{2.35}$$

Then the following Corollary holds.

Corollary 2. 1 *Vectors* $\mathbf{x}^* \in \mathbb{R}^n$, $\boldsymbol{\alpha}^* \in \mathbb{R}^m$ *satisfy* $\mathbf{x}^* = P_c(D, \boldsymbol{\alpha}^*, \mathbf{a}, \mathbf{b})$ *if and only if there exists* $\mathbf{t}^* \in \mathbb{R}^m$ *such that*

$$\mathbf{x}(\mathbf{t}^*) = \mathbf{x}^* \tag{2.36}$$

$$\boldsymbol{\alpha}(\mathbf{t}^*) = \boldsymbol{\alpha}^*. \tag{2.37}$$

Proof of Corollary 2.1 follows from the statement of problem $(C_m^=)$ and (2.27).

From Corollary 2.1 it follows that $\mathbf{x}^* = P_c(D, \boldsymbol{\alpha}^*, \mathbf{a}, \mathbf{b})$ can be solved with respect to $\mathbf{x}^*$ for given $\boldsymbol{\alpha}^*$ by first solving (2.37) for $\mathbf{t}^*$ and then calculating $\mathbf{x}^*$ by using (2.36).

Let S be the set of solutions to (2.37) for a particular value of $\boldsymbol{\alpha}^*$:

$$S = \{\mathbf{t} \in \mathbb{R}^m : \boldsymbol{\alpha}(\mathbf{t}) = \boldsymbol{\alpha}^*\}. \tag{2.38}$$

According to (2.35), each component of $\boldsymbol{\alpha}(\mathbf{t})$ is a linear combination of the same set of terms. Each term $(\mathbf{c}')^{-1}\Big[-D^T\mathbf{t}\Big]_{\mathbf{c}_j'(a_j)}^{\mathbf{c}_j'(b_j)}$ is a smooth function of $\mathbf{t}$ except on the pair of break hyperplanes

$$A_j = \{\mathbf{t} \in \mathbb{R}^m : \langle \mathbf{D}_j, \mathbf{t}\rangle = -c_j'(a_j)\}, \tag{2.39}$$

$$B_j = \{\mathbf{t} \in \mathbb{R}^m : \langle \mathbf{D}_j, \mathbf{t}\rangle = -c_j'(b_j)\}. \tag{2.40}$$

In the case when $m = 1$, that is, when there is a single linear equality constraint of the form (1.5) in problem $(C_m^=)$, the break hyperplanes are reduced to break points.

3 Illustrative examples

In this section, we illustrate application of Theorem 2.1 and Theorem 2.2 to simple particular problems.

Example 1. Solve the problem

$$\min \Big\{c(\mathbf{x}) = x_1^3 + x_2^3\Big\}$$

subject to

$$x_1^2 + 2x_2^2 \le 10$$

$$1 \le x_1 \le 3$$

$$1 \le x_2 \le 5.$$

This problem is of the form (SCS) with $c_j(x_j) = c_j x_j^q$, $q = 3, c_1 = 1, c_2 = 1, p = 2, d_1 = 1, d_2 = 2, \alpha = 10, a_1 = 1, b_1 = 3, a_2 = 1, b_2 = 5$. Objective function $c(\mathbf{x}) = x_1^3 + x_2^3$ is strictly convex for the feasible $\mathbf{x} = (x_1, x_2) \ge (1, 1)$, and constraint function $d(\mathbf{x}) = x_1^2 + 2x_2^2$ is strictly convex for every $\mathbf{x} \in \mathbb{R}^2$.

The optimal solution, obtained by Theorem 2.1, is

$$\mathbf{x}^* = (x_1^*,\ x_2^*) = (1,\ 1),$$

and

$$c_{\min} = c(\mathbf{x}^*) = 2.$$

Example 2. Solve the problem

$$\min \left\{ c(\mathbf{x}) = 4x_1^2 + 10x_2^2 + 4x_3^2 + 3x_4^2 + 7x_5^2 + 3x_6^2 + x_7^2 \right\}$$

subject to

$$\sum_{j=1}^{7} x_j = 72$$

$$4 \leq x_1 \leq 7$$

$$4.5 \leq x_2 \leq 10$$

$$8 \leq x_3 \leq 13$$

$$5 \leq x_4 \leq 8$$

$$4 \leq x_5 \leq 7$$

$$30 \leq x_6 \leq 40$$

$$4 \leq x_7 \leq 7.$$

This problem is of the form $(C_m^=)$ with $m = 1$, $n = 7$; $c_j(x_j) = c_j x_j^q$, $q = 2$; $\alpha = 72$; $d_j = 1$, $j = 1, \dots, 7$; $\mathbf{c} = (c_j)_{j=1}^7 = (4, 10, 4, 3, 7, 3, 1)$,

$$\mathbf{a} = (a_j)_{j=1}^7 = (4,\ 4.5,\ 8,\ 5,\ 4,\ 30,\ 4), \quad \mathbf{b} = (b_j)_{j=1}^7 = (7, 10, 13, 8, 7, 40, 7).$$

The optimal solution is

$$\mathbf{x}^* = (7,\ 4.5,\ 9.8636,\ 8,\ 5.6363,\ 30,\ 7),$$

and

$$c_{\min} = c(\mathbf{x}^*) = 3951.0454.$$

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